UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M14A: Mathematical Methods 1

COURSE CODE	:	**MATHM14A**
UNIT VALUE	:	**0.50**
DATE	:	**11-MAY-06**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

06-C1041-3-210

TURN OVER

All questions may be answered, but only marks obtained on the best **four** *questions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

1. (a) Define the scalar product $\mathbf{a} \cdot \mathbf{b}$ and the vector product $\mathbf{a} \wedge \mathbf{b}$ of the two vectors $\mathbf{a}$ and $\mathbf{b}$.

   (b) Show that the volume of the parallelepiped with three concurrent (i.e. sharing a common corner) edges, given by the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, is $|[\mathbf{a}, \mathbf{b}, \mathbf{c}]|$ where $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = (\mathbf{a} \wedge \mathbf{b}) \cdot \mathbf{c}$.

   (c) Show that

$$(\mathbf{a} \wedge \mathbf{b}) \wedge (\mathbf{c} \wedge \mathbf{d}) = [\mathbf{a}, \mathbf{c}, \mathbf{d}]\mathbf{b} - [\mathbf{b}, \mathbf{c}, \mathbf{d}]\mathbf{a} = [\mathbf{d}, \mathbf{a}, \mathbf{b}]\mathbf{c} - [\mathbf{c}, \mathbf{a}, \mathbf{b}]\mathbf{d},$$

   and hence find a representation of any vector in terms of three other vectors that are not themselves coplanar.

2. (a) If $z = x + iy$, express the following constraints on z as constraints on x and y.

$$\text{i)} \quad |z - i| = |z + 1|, \qquad \text{ii)} \quad |z - i| \geq |z + i|.$$

   (b) Show that

$$1 + z + z^2 + z^3 + z^3 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}$$

   for positive integer n. By considering the limit $n \to \infty$, show that if a and θ are real with $|a| < 1$ then,

$$a \sin\theta + a^2 \sin 2\theta + a^3 \sin 3\theta + \cdots = \frac{a \sin\theta}{1 - 2a\cos\theta + a^2}.$$

3. (a) Show that the function

$$y(x) = \frac{\exp(ax)}{1+\exp(x)},$$

has a stationary point at

$$x = \ln\left(\frac{a}{1-a}\right), \quad y = a^a(1-a)^{1-a},$$

for $0 < a < 1$. Sketch the graph of the function for the three cases $a = \frac{1}{2}$, $a = 1$, $a = 2$ on a single set of axes.

(b) *You are given that* the function

$$y(x) = \sinh^{-1} x + (\sinh^{-1} x)^2,$$

satisfies the differential equation

$$(1+x^2)y'' + xy' = 2,$$

where a prime denotes differentiation with respect to x. Use Leibnitz' rule for differentiating a product to show that, if n is even,

$$y^{(n)}(0) = -(-1)^{\frac{n}{2}}(n-2)^2(n-4)^2\cdots(2)^2(2),$$

where $y^{(n)}(x)$ is the nth derivative of $y(x)$. What is the equivalent expression for odd values of n?

4. Find the integrals

a) $\int_0^\infty x\exp(-x)\mathrm{d}x$, b) $\int_{-\infty}^1 \frac{\exp(x)}{1+2\exp(x)}\,\mathrm{d}x$, c) $\int_0^1 x\tan^{-1}(x)\,\mathrm{d}x$,

d) $\int \frac{\tan(\ln x)}{x}\mathrm{d}x$, e) $\int_1^\infty \frac{\mathrm{d}x}{x^3+x^2+x}$.

5. Find the solution of the equations

(a) $y' = 2 + y^2$, $y(0) = 0$.

(b) $y' + y\sin x = x\exp(\cos x)$, $y(0) = 0$,

(c) $4(2x^2 + xy)y' = 3y^2 + 4xy$.

where a prime denotes differentiation with respect to x.

6. Solve the differential equations

(a) $x^2y'' - 2xy' + 2y = 1$, $y(1) = 1$, $y'(1) = 0$,

(b) $y'' + y' - 6y = x + \exp(2x)$, $y(0) = 0$, $y'(0) = 1/5$.

where a prime denotes differentiation with respect to x.

END OF PAPER