UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M14A: Mathematical Methods 1

COURSE CODE : **MATHM14A**

UNIT VALUE : **0.50**

DATE : **29-APR-05**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

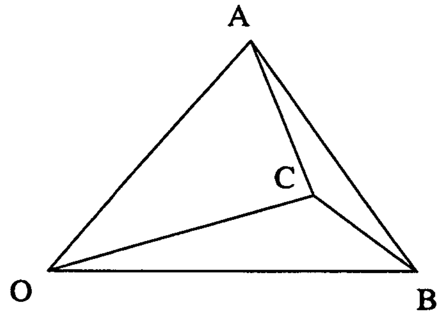
05-C0959-3-210

TURN OVER

All questions may be answered, but only marks obtained on the best **four** *questions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

1. (a) Define the scalar product $\mathbf{a} \cdot \mathbf{b}$ of the vectors $\mathbf{a}$ and $\mathbf{b}$. Give a full definition of their vector product $\mathbf{a} \wedge \mathbf{b}$.

   (b) A tetrahedron is formed by the four points O, A, B, C.



   If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the position vectors of A, B and C relative to O, show that the vector from the midpoint of OA to the midpoint of the opposite side, CB, is

$$\tfrac{1}{2}(\mathbf{b} + \mathbf{c} - \mathbf{a}).$$

   Show that the lines joining pairs of opposite sides meet and that, if all the edges of the tetrahedron have the same length, then they do so at right angles.

2. State De Moivre's theorem and use it to show that

   (a)
$$8(1 + i\sqrt{3})^{-3} + 1 = 0.$$

   (b)
$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}.$$

   Hence show that the roots of the equation

$$x^4 + 4x^3 - 6x^2 - 4x + 1 = 0,$$

   are

$$\tan\left(\frac{\pi}{16} + n\frac{\pi}{4}\right), \quad n = 0, 1, 2, 3.$$

3. Sketch the functions $\sinh^{-1} x$, $\cosh^{-1} x$, $\tanh^{-1} x$ on the same graph. Establish, from the definition of $\sinh x$, that

$$\sinh^{-1} x = \ln\left(x + \sqrt{1+x^2}\right).$$

You are given that $y = (\sinh^{-1} x)^2$ satisfies the equation

$$(1+x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} = 2.$$

Use Leibnitz theorem to show that

$$\frac{\mathrm{d}^{n+2} y}{\mathrm{d}x^{n+2}}(0) + n^2\frac{\mathrm{d}^n y}{\mathrm{d}x^n}(0) = 0,$$

and deduce the first four non-zero terms in the power series expansion of $y(x)$ about $x = 0$.

4. (a) Show that

$$\frac{1}{\sin\theta\cos\theta} = \frac{\sec^2\theta}{\tan\theta},$$

and *hence* deduce that

$$\int \frac{\mathrm{d}x}{\sin x} = \ln\left(\tan(x/2)\right) + \text{ constant.}$$

Find the solution of the differential equation

$$\sin x\frac{\mathrm{d}y}{\mathrm{d}x} + y = \sin^2 x,$$

that is finite at $x = 0$.

(b) Use the substitution $t = \tan(\theta/2)$ to show that

$$\int_{\pi/2}^{\pi} \frac{\mathrm{d}\theta}{1 + \sin\theta - \cos\theta} = \ln 2.$$

5. Solve the differential equations

(a) $(y - x - 4)y' = (y - x + 2)$,

(b) $xyy' = y^2 - \sqrt{x^2 + y^2}, \quad y(1) = 0.$

where primes denotes differentiation with respect to x.

6. Solve the differential equations

(a) $y'' - 7y' + 6y = \cosh x + x$,

(b) $x^2 y'' + 4xy' + 2y = 3/x^2, \quad y(1) = y'(1) = 1.$

where primes denotes differentiation with respect to x.

END OF PAPER