UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M14A: Mathematical Methods 1

COURSE CODE	:	**MATHM14A**
UNIT VALUE	:	**0.50**
DATE	:	**13-MAY-04**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

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TURN OVER

All questions may be answered, but only marks obtained on the best **four** *questions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

1. (a) If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three vectors.
   (i) Define the scalar product $\mathbf{a} \cdot \mathbf{b}$,
   (ii) Give a careful definition of the vector product $\mathbf{a} \wedge \mathbf{b}$,
   (iii) Define the scalar triple product $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$.

   (b) Show that a plane containing the origin, the x-axis and a vector with direction cosines (p, q, r) has Cartesian equation.

$$ry = qz.$$

   (c) Find the acute angle between the line $\mathbf{r} = \mathbf{i} - 2\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} - 3\mathbf{k})$ and the plane $\mathbf{r}.(2\mathbf{i} - \mathbf{j} + 4\mathbf{k}) = 10$. How far is the plane from the point $(1, 0, 0)$?

2. (a) If z is the complex number $x + iy$, show that

$$\frac{1}{z} = \frac{x - iy}{x^2 + y^2}.$$

   (b) Recall that $\exp(i\theta) = \cos\theta + i\sin\theta$ for real values of θ. Show

$$\frac{1}{1 + \exp(i\theta)} = \frac{1}{2} - \frac{i}{2}\tan(\theta/2)$$

   and describe the path followed by the point $w = 1/(1 + z)$ as z starts at $z = 1$ and moves, in an anticlockwise direction, around the unit circle in the Argand diagram.

   (c) Use complex numbers to evaluate the integral

$$\int_0^\pi \exp(-ax)\sin(nx)\,\mathrm{d}x,$$

   for integer valuies of n. Comment on the limit $a \to \infty$.

3. (a) Write down the Maclaurin expansions (i.e. about $x = 0$) of $\sin x$ and $\cos x$ clearly giving the general term in the expansions.

   (b) Consider the function

$$f(x) = \frac{\sin x}{b + \cos ax}.$$

   (i) Show the nth derivative $f^{(n)}(0) = 0$ if n is an even integer,

   (ii) Find a relation between a and b that ensures $f^{(3)}(0) = 0$.

   (c) The function $y(x)$ satisfies the equations

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - xy = 0, \quad y(0) = 1, \quad y^{(1)}(0) = 0.$$

   Use Leibniz theorem for differentiating a product to show that

$$y^{(n)}(0) = (n-2)y^{(n-3)}(0), \quad n > 2.$$

   Hence write down the first 5 non-zero terms in the Maclaurin expansion of $y(x)$ about $x = 0$.

4. (a) Evaluate the following integrals.

$$\int_0^\infty \frac{2\,\mathrm{d}x}{(x+1)(x^2+1)}\ ; \qquad \int_0^a \sinh^{-1} x\,\mathrm{d}x.$$

   (b) If $t = \tanh x$, show that

$$\cosh^2 x = \frac{1}{1-t^2} \quad \text{and} \quad \sinh^2 x = \frac{t^2}{1-t^2}.$$

   Use the substitution $t = \tanh x$ to evaluate

$$\int \frac{\mathrm{d}x}{1 + a\sinh^2 x}, \qquad a > 1.$$

5. (a) Solve the equations

$$(i)\quad x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = x^3 \ln x, \quad y(1) = -1, \qquad (ii)\quad x^2\frac{\mathrm{d}y}{\mathrm{d}x} = xy + y^2, \quad y(1) = 1.$$

   (b) Use the transformation $u = y^{-2}$ to solve the equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = x^3 y^3.$$

6. Solve the differential equations

   (a) $y'' - 4y' + 4y = \cosh ax, \quad a \neq 2,$

   (b) $x^2 y'' + xy' + 4y = \ln x, \quad x > 0, \quad y(1) = y'(1) = 1$

   where a dash ($'$) denotes differentiation with respect to x.

END OF PAPER