UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M14A: Mathematical Methods 1

COURSE CODE	:	**MATHM14A**
UNIT VALUE	:	**0.50**
DATE	:	**07-MAY-03**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

03-C0946-3-190

TURN OVER

All questions may be answered, but only marks obtained on the best **four** *questions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

1. (a) Define the scalar product $\mathbf{a} \cdot \mathbf{b}$ of the vectors $\mathbf{a}$ and $\mathbf{b}$. Give a careful definition of their vector product $\mathbf{a} \wedge \mathbf{b}$.

   (b) Show that the distance between the two skew lines $\mathbf{r}_1 = \mathbf{a}_1 + \lambda \mathbf{b}_1$ and $\mathbf{r}_2 = \mathbf{a}_2 + \mu \mathbf{b}_2$ is

$$\left| \frac{(\mathbf{a}_1 - \mathbf{a}_2) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2)}{|(\mathbf{b}_1 \wedge \mathbf{b}_2)|} \right|.$$

   (c) Show that if the two lines

$$\frac{x - c_1}{d_1} = \frac{y - c_2}{d_2} = \frac{z - c_3}{d_3},$$

$$\frac{x - d_1}{c_1} = \frac{y - d_2}{c_2} = \frac{z - d_3}{c_3},$$

   intersect, they lie in the plane

$$\mathbf{r} \cdot (\mathbf{c} \wedge \mathbf{d}) = 0,$$

   where $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, $\mathbf{d} = d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{k}$ and $\mathbf{c}$ and $\mathbf{d}$ are not parallel vectors.

2. (a) State De Moivre's theorem and use it to show that if $z = \exp(i\theta)$ then

$$(z^n + z^{-n}) = 2\cos(n\theta) \quad \text{and} \quad (z^n - z^{-n}) = 2i\sin(n\theta),$$

   where $n \geq 0$ is an integer.

   (b) Hence show that

$$\sin^5\theta = \tfrac{1}{16}\left(\sin(5\theta) - 5\sin(3\theta) + 10\sin(\theta)\right).$$

   (c) Hence, or otherwise, evaluate

$$\int_0^{\pi} \theta \sin^4(\theta)\cos(\theta)\, \mathrm{d}\theta.$$

3. The function defined by

$$\mathrm{gd}(t) = \tan^{-1}(\sinh t)$$

is known as the *gudermannian* of t.

(a) Show that gd is odd and sketch its graph.

(b) Given that

$$\tan^{-1}(u) = u - \tfrac{1}{3}u^3 + \tfrac{1}{5}u^5 + \cdots,$$
$$\sinh t = t + \tfrac{1}{6}t^3 + \tfrac{1}{120}t^5 + \cdots,$$

find the power series for $\mathrm{gd}(x)$ up to and including terms in x^5.

(c) Show, directly from the definition, that

$$\mathrm{gd}(t) = \int_0^t \operatorname{sech} u \, \mathrm{d}u.$$

4. (a) Evaluate

$$\int_0^\infty \frac{x \, \mathrm{d}x}{(x+1)^2(x^2+1)}, \quad \int_0^\infty \frac{\tan^{-1}(x)}{1+x^2} \, \mathrm{d}x.$$

(b) If

$$I_n = \int_{\pi/2}^x \frac{\cos^{2n+1} t}{\sin t} \, \mathrm{d}t, \quad n \geq 0,$$

show that

$$2(n+1)I_{n+1} = 2(n+1)I_n + \cos^{2n+2} x, \quad n > 0.$$

Hence find I_3.

5. (a) Find the solution of

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + (x-2)y = x^4, \quad y(1) = 1.$$

(b) Use the substitution $z = \tan y$ to find the general solution of the differential equation

$$\sec^2 y \frac{\mathrm{d}y}{\mathrm{d}x} + 2x \tan y = x.$$

6. Solve the differential equations

(a) $y'' + y' - 12y = \cosh 3x$,

(b) $x^2y'' - 2xy' + 2y = \ln x, \quad y(1) = y'(1) = 1.$

where a dash denotes differentiation with respect to x.