UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sc.* *M.Sci.*

Mathematics C348: Mathematics For General Relativity

COURSE CODE : **MATHC348**

UNIT VALUE : **0.50**

DATE : **17-MAY-05**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

05-C0931-3-70

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

Geodesic Equation ($U^a = \mathrm{d}X^a/\mathrm{d}\tau$)

$$\begin{aligned}
\frac{\mathrm{d}U^a}{\mathrm{d}\tau} + \Gamma^a{}_{bc}\, U^b U^c &= 0; &(1)\\
\frac{\mathrm{d}p_a}{\mathrm{d}\tau} &= \frac{m}{2}\left(\partial_a g_{bc}\right) U^b U^c; &(2)\\
\Gamma^a{}_{bc} &= \frac{1}{2} g^{ad}\left(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}\right). &(3)
\end{aligned}$$

Geodesics parameterized by λ

$$\frac{\mathrm{d}^2 X^a}{\mathrm{d}\lambda^2} + \Gamma^a{}_{bc}\frac{\mathrm{d}X^b}{\mathrm{d}\lambda}\frac{\mathrm{d}X^c}{\mathrm{d}\lambda} = 0. \qquad (4)$$

Schwarzschild metric line element

$$\mathrm{d}\tau^2 = \left(1 - \frac{r_s}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{r_s}{r}\right)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2); \qquad r_s = 2GM.$$

Faraday tensor

$$F_{ab} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}; \qquad F^{ab} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}.$$

Dual Faraday tensor

$$^*F^{ab} \equiv \frac{1}{2}\,\epsilon^{abcd}F_{cd} = \begin{pmatrix} 0 & +B_x & +B_y & +B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{pmatrix}.$$

Maxwell Source Equations

$$\partial_b F^{ab} = j_e^a.$$

Internal Maxwell Equations

$$\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0 \qquad \text{or} \qquad \partial_b {}^*F^{ab} = 0.$$

1. (a) Determine which of the following equations is a valid tensor equation. Describe the errors in the other equations.

   (i) $A_{ab} = B_a + C_b + D_{ab}$;

   (ii) $M^{ab} = G^{ba} + Q^a R^b$;

   (iii) $J_b = T_a{}^c F_{bc} Z^{ad}{}_d$;

   (iv) $K^c{}_b = Y_a Z^{ac}{}_a X^a{}_b$;

   (b) Describe how $V^a W_b$ transforms from unprimed coordinates X^a to primed coordinates X'^a. Show that $V^a W_a$ is a scalar.

   (c) Consider a two-dimensional manifold M with coordinates X^1 and X^2. Suppose the tensors V^a, T_{ab}, and $P^{ab}{}_c$ have the values

$$V: \qquad V^1 = 2; \qquad V^2 = 3;$$

$$T: \quad \begin{pmatrix} T_{11} = 2 & T_{12} = 3 \\ T_{21} = 1 & T_{22} = 4 \end{pmatrix};$$

$$P: \quad \begin{pmatrix} P^{11}{}_1 = 1 & P^{12}{}_1 = 2 \\ P^{21}{}_1 = 1 & P^{22}{}_1 = 0 \end{pmatrix}; \qquad \begin{pmatrix} P^{11}{}_2 = 4 & P^{12}{}_2 = 0 \\ P^{21}{}_2 = 1 & P^{22}{}_2 = 5 \end{pmatrix}$$

   Find the following tensors:

   (i) $Q^{bc} = V^b V^c$;

   (ii) $R^a{}_b = T_{bc} Q^{ac}$;

   (iii) $S^a = P^{ba}{}_b$.

   (d) Suppose S^{ab} is a symmetric tensor and A_{ab} is antisymmetric. Derive a simple expression for

$$S^{ab} A_{ab},$$

   showing your work.

   (e) Using index notation prove the vector identity

$$\nabla \times \nabla f = 0.$$

2. Suppose a spaceship rest frame S moves at a velocity V in the x direction with respect to the Earth frame E. Let U^a, m, and $\vec{\mathbf{p}}$ be the 4-velocity, rest mass, and 3-momentum of the spaceship. Also let the 4-acceleration be $a^a = dU^a/d\tau$.

(a) (i) What is U^a_{S}? What is U^a_{E}?

(ii) Show that $\overline{\mathbf{U}} \cdot \overline{\mathbf{U}} = 1$.

(iii) Show that $\overline{\mathbf{U}} \cdot \overline{a} = 0$.

(iv) Define the energy–momentum form with components E and $-\vec{\mathbf{p}}$ and show that $E^2 = |\vec{\mathbf{p}}|^2 + m^2$.

(b) A metre stick is at rest in the spaceship rest frame S. Derive an expression for its length as measured in the Earth frame E. What is its length as measured in E if $V = 0.8$?

(c) Suppose the spaceship passes through an electro-magnetic field described by the Faraday tensor F_{ab} and the dual Faraday tensor ${}^*F^{ab}$. Define a four vector

$$B^b = U_a {}^*F^{ab}$$

where $U_a = g_{ac}U^c$. Express ${B_{\mathsf{E}}}^b$ and ${B_{\mathsf{S}}}^b$ in terms of $\vec{\mathbf{E}}$, $\vec{\mathbf{B}}$, V, and $\gamma = (1 - V^2)^{-1/2}$.

3. (a) Consider a particle of mass m_0 moving in a geodesic around an object of mass M whose gravitational field is described by the Schwarzschild metric. The particle moves in the plane $\theta = \pi/2$. Rewrite the Schwarzschild metric in the coordinates (t, x, ϕ), where $x = r/r_s$.

(b) Show that along a geodesic there are two conserved quantities k and h. Express $\mathrm{d}t/\mathrm{d}\tau$ and $\mathrm{d}\phi/\mathrm{d}\tau$ in terms of k and h.

(c) Derive an expression for $\mathrm{d}x/\mathrm{d}\tau$ in the form

$$\frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 + V(x) = C$$

where C is a constant. What are the effective potential $V(x)$ and the effective energy C?

(d) Suppose that $h = 2r_s$. Find the radii x_1 and x_2, $x_1 < x_2$ where the effective potential has an extremum (maximum or minimum). Show that the particle can have circular orbits at these radii for particular values of C. What is the value of C for a circular orbit at x_1?

(e) Give a rough sketch of $V(x)$ (again with $h = 2r_s$).

(f) Show from the sketch, or otherwise, that the outer circular orbit at x_2 is stable. Also show that the inner orbit at x_1 is unstable.

4. The metric line element for Euclidean 3-space E^3 is $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$. The two-dimensional surface H^2 embedded in E^3 is parameterized by coordinates $(X^1, X^2) = (r, \phi)$, where

$$\begin{aligned} x &= R\cosh r\cos\phi \\ y &= R\cosh r\sin\phi \\ z &= R\sinh r. \end{aligned}$$

   (a) Express the metric line element in the coordinates (r, ϕ). What is the metric tensor g_{ab} in these coordinates?

   (b) Calculate the Christoffel symbols $\Gamma^a{}_{bc}$ for this metric.

   (c) From the geodesic equation, find the differential equations for a geodesic, i.e. find $\mathrm{d}^2r/\mathrm{d}s^2$ and $\mathrm{d}^2\phi/\mathrm{d}s^2$ for a geodesic parameterized by s.

5. Let f be a scalar field and X^a be a vector field. The covariant derivative ∇_b acts on these fields as follows:

$$\begin{aligned} \nabla_b f &= \partial_b f \\ (\nabla_b X)^a &= \partial_b X^a + \Gamma^a{}_{bc}X^c. \end{aligned}$$

Also, ∇_b obeys the product rule.

   (a) In general relativity, explain what is meant by a locally inertial frame (LIF) near a spacetime event P? What two conditions on the metric g_{ab} apply in an LIF? What values do the Christoffel symbols $\Gamma^a{}_{bc}$ take in an LIF?

   (b) Derive the formula for the derivative of a type $\binom{0}{2}$ second rank tensor M_{cd}:

$$(\nabla_b M)_{cd} = \partial_b M_{cd} - \Gamma^a{}_{bc}M_{ad} - \Gamma^a{}_{bd}M_{ac}.$$

   (c) Let g_{cd} be the metric tensor. Calculate $(\nabla_b g)_{cd}$, showing your work.

END OF PAPER