UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. *M.Sci.*

Mathematics C348: Mathematics For General Relativity

COURSE CODE	:	**MATHC348**
UNIT VALUE	:	**0.50**
DATE	:	**20-MAY-02**
TIME	:	**14.30**
TIME ALLOWED	:	**2 hours**

02-C0916-3-70

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

Geodesic Equation ($u^a = dx^a/d\tau$):

$$\frac{du^a}{d\tau} + \Gamma^a{}_{bc}\, u^b u^c = 0; \tag{1}$$

$$\frac{du_a}{d\tau} = \frac{1}{2}(\partial_a g_{bc})\, u^b u^c; \tag{2}$$

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad}(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}). \tag{3}$$

Schwarzschild metric line element:

$$d\tau^2 = \left(1 - \frac{r_s}{r}\right) dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2); \qquad r_s = 2GM.$$

Faraday tensor:

$$F_{ab} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \qquad F^{ab} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}.$$

Dual Faraday tensor:

$$\mathfrak{F}^{\ ab} = \begin{pmatrix} 0 & +B_x & +B_y & +B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{pmatrix}.$$

Maxwell Source Equations:

$$\partial_b F^{ab} = j^a.$$

Internal Maxwell Equations:

$$\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0.$$

1. (a) Describe how the metric tensor g_{bc} transforms from unprimed coordinates x^a to primed coordinates x'^a.

(b) Suppose that in some coordinate system the tensor $\delta^a{}_b$ has the form

$$\delta^a{}_b = \begin{cases} +1 & a = b \\ 0 & a \neq b \end{cases}.$$

Show that it also has this form in any other coordinate system.

(c) Consider a two-dimensional manifold M with coordinates x^1 and x^2. Suppose the tensors $A^{ab}{}_c$, B_{ab}, and C^a have the values

$$\begin{array}{cccc} A^{11}{}_1 = -2, & A^{11}{}_2 = 4, & A^{12}{}_1 = -1, & A^{12}{}_2 = 0, \\ A^{21}{}_1 = 3, & A^{21}{}_2 = 3, & A^{22}{}_1 = 2, & A^{22}{}_2 = 5, \\ B_{11} = 1, & B_{12} = 0, & B_{21} = 1, & B_{22} = -2, \\ C^1 = 0, & C^2 = -2. & & \end{array}$$

Also the metric has the values

$$g_{11} = 3, \qquad g_{12} = 5, \qquad g_{22} = -3.$$

Find the following tensors.

(i) $P_b = B_{ab}C^a$

(ii) $f = C^c A^{ab}{}_c B_{ab}$

(iii) $Q^a = A^{ab}{}_b$

(iv) C_b

(d) Only one of the following equations is a valid tensor equation. Determine which equation is valid, and describe the errors in the other equations.

(i) $D_{ab} = T_a W^a{}_b$

(ii) $E_{ab} = F^c{}_{ba} C_c + L^d S_{dba}$

(iii) $Z_{mn} = Y^a{}_m{}^a{}_n$

(iv) $P_c = K^a{}_a R_c J^a{}_a$

(e) Using index notation prove the vector identity

$$\nabla \times (f \nabla g) = \nabla f \times \nabla g.$$

2. Consider the surface of the Earth (assumed to be perfectly spherical). In terms of its radius R, co-latitude θ and longitude ϕ, the metric line element is

$$ds^2 = R^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

(a) Suppose a ship at some position (θ, ϕ) travels with compass bearing ψ (here $\psi = 0$ if the ship is heading due North, $\psi = \pi/4$ if the ship is heading Northeast, and so on). If the ship travels a small distance, with a change $\delta\theta$ in co-latitude and a change $\delta\phi$ in longitude, what is the ratio $\delta\theta/\delta\phi$?

(b) A Mercator map projection uses coordinates (x, y), where the coordinate transformations are

$$\begin{aligned} x &= \frac{w}{2\pi}\phi, \\ y &= a \log \cot \frac{\theta}{2} \end{aligned}$$

with w, a constants. Find $\delta y/\delta x$ in terms of $\delta\theta/\delta\phi$. Consider a direction on the Mercator map which makes an angle of $\tilde{\psi}$ with respect to the vertical. Show that $\tilde{\psi} = \psi$ only if $a = w/(2\pi)$.

(c) Find the metric line element ds^2 for the Mercator coordinates x and y with $a = w/(2\pi)$ (hint: you may use the identity $\sin\theta = 1/\cosh(\log\cot\frac{\theta}{2})$ without proof).

(d) What is the range of the Mercator coordinate x? What is the range of the Mercator coordinate y? Briefly explain why Greenland appears to be larger than Africa on a Mercator map, even though in reality it is much smaller.

(e) A polar map projection uses coordinates $(x^1, x^2) = (\rho, \lambda)$ where

$$\rho = R\sin\theta, \lambda = \phi.$$

What is the metric in these coordinates?

(f) Find the Christoffel symbols $\Gamma^2{}_{bc}$ for the metric in (e), i.e. find

$$\begin{pmatrix} \Gamma^2{}_{11} & \Gamma^2{}_{12} \\ \Gamma^2{}_{21} & \Gamma^2{}_{22} \end{pmatrix}.$$

3. For this question, assume Special Relativity holds (i.e. flat space with $g_{ab} = \eta_{ab}$).

(a) Suppose a particle is moving with 3-velocity $\vec{V}$ relative to some inertial frame. What is its 4-velocity u^a? What is u_b?

(b) Write down the Maxwell Source Equations for $a = 0, 1, 2, 3$, expressing these equations in terms of the electric and magnetic fields and the charge density and current.

(c) Show from the Maxwell Source Equation that electric charge is conserved.

(d) Consider a charged particle with mass m, 4-momentum (in lowered co-vector form) $p_b = mu_b$, and charge q. The particle travels through an electromagnetic field with Faraday tensor F_{ab}. The Lorentz force on the particle is

$$\frac{dp_b}{d\tau} = qu^a F_{ab}$$

What is $dp_0/d\tau$ in terms of $\vec{E}$ and $\vec{B}$? What is $dp_1/d\tau$? Also find dp^0/dt and dp^1/dt.

(e) Show that the Lorentz force 4-vector is perpendicular to the velocity 4-vector, i.e. show that

$$\frac{dp_b}{d\tau}u^b = 0.$$

(f) Suppose an inertial reference frame B moves at a velocity $V\hat{x}$ with respect to inertial frame A. Consider the Lorentz transformation from A to B given by

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}_B = \begin{pmatrix} \gamma & -\gamma V & 0 & 0 \\ -\gamma V & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}_A$$

where $\gamma = (1 - V^2)^{-1/2}$.

How do the electric field components E_x, E_y, E_z transform under this Lorentz transformation (show your derivation)?

4. (a) Consider a particle of mass m moving in a geodesic around an object of mass M whose gravitational field is described by the Schwarzschild metric. The particle moves in the plane $\theta = \pi/2$. The Schwarzschild metric in this plane has two symmetries. What are they? Show that the particle's orbit has two conserved quantities h and k.

(b) Derive an expression for $dr/d\tau$ in the form

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V(r) = C$$

where C is a constant, and the effective potential is

$$V(r) = -\frac{1}{2}\left(\frac{r_s}{r} - \frac{h^2}{r^2} + \frac{r_s h^2}{r^3}\right).$$

What is the effective energy C?

(c) Find the radii r_1 and r_2, $r_1 < r_2$ where the effective potential has an extremum (maximum or minimum). Show that if $C = V(r_1)$ or $C = V(r_2)$ then the requirements for a circular orbit ($dr/d\tau = d^2r/d\tau^2 = 0$) are satisfied. Show that $h \geq \sqrt{3}\, r_s$ for these orbits. Also show that $r_2 \geq 3r_s$.

(d) Let $h = 2r_s$. What are r_1 and r_2? Show that for the outer orbit at r_2, $V''(r_2) > 0$ and hence that this orbit is stable. Is the inner orbit at r_1 stable?

5. (a) Consider a spatial manifold. Let $u^a(s) = dx^a/ds$ be the tangent to a curve parameterized by arclength s. The geodesic equation is

$$\frac{du^a}{ds} + \Gamma^a{}_{bc}\, u^b u^c = 0,$$

where $d/ds = u^a \partial_a$. From this form of the geodesic equation, derive the second form

$$\frac{du_a}{ds} = \frac{1}{2}(\partial_a g_{bc}).u^b u^c.$$

(b) Consider a sphere of radius 1 as a 2-dimensional manifold with coordinates $x_1 = \theta$, $x_2 = \phi$, and line-element

$$ds^2 = \left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

Show that geodesics have a conserved quantity (call it H).

(c) Using the metric line element, or otherwise, derive an equation for $d\theta/ds$ in terms of H.

(d) Let $X = \cos\theta$. Show that X satisfies

$$\left(\frac{dX}{ds}\right)^2 = 1 - X^2 - H^2.$$

(e) Obtain a second-order differential equation for $X(s)$ and write down its general solution. Suppose a geodesic starts at co-latitude θ_0 heading due East. Find $X(s)$ and hence $\theta(s)$ explicitly in terms of θ_0. Show that the total length of the geodesic (i.e. the length needed to go all the way around the sphere once) is independent of θ_0.