UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:–

B.Sc. M.Sci.

Mathematics C381: Logic

COURSE CODE : **MATHC381**

UNIT VALUE : **0.50**

DATE : **29-APR-05**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

05-C0937-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Give the definition of a propositional language L and the set of L-formulas.
   (b) Give the definition of the formation tree of an L-formula and give the unique formation tree for the L-formula

$$(((p \Rightarrow q)\&(q \vee r)) \Rightarrow (p \vee r)) \Rightarrow \neg(q \vee s)$$

   (c) State what it means for an L-formula to be in disjunctive normal form, and use the truth table method to put $(p \Rightarrow q) \Rightarrow r$ in disjunctive normal form.

2. (a) State and prove the Soundness Theorem for the Predicate Calculus. You may assume that if τ and τ' are tableaux such that τ' is an immediate extension of τ and τ is satisfiable, then τ' is satisfiable.
   (b) Use the Soundness Theorem for an unsigned tableau to show that

$$(\exists x(Px \vee Qx)) \Leftrightarrow ((\exists x Px) \vee (\exists x Qx))$$

   is logically valid.

3. (a) Give the definition of primitive recursive function, partial recursive function and recursive function.
   (b) Give the definitions of Turing machine, Turing program and Turing computable partial function.
   (c) Disjoint subsets A and B of $\mathbb{N}$ are said to be recursively inseparable if there is no recursive set C such that $A \subset C$ and $C \cap B = \emptyset$. Show that $A = \{x : \phi_x(x) = 0\}$ and $B = \{x : \phi_x(x) = 1\}$ are disjoint recursively enumberable sets which are recursively inseparable.
   (d) For A as in part c), prove that $K \equiv_1 A$, where $K = \{x \in \mathbb{N} : \phi_x(x) \text{ converges}\}$.

4. By arithmetic we mean the set of sentences which are true in the structure

$$(\mathbb{N}, +, \cdot, 0, 1, =).$$

(a) Give the definition of an arithmetical definable k-place partial function.

(b) For each formula F in the language of arithmetic let $\sharp(F)$, be the Gödel number of F. If $\mathcal{F}$ is any collection of formulas, $\sharp\mathcal{F}$ denotes $\{\sharp F : F \in \mathcal{F}\}$. Let $TrueSnt$ $= \sharp\{\text{all true sentences}\}$. Show that every arithmetically definable set $A \subset \mathbb{N}$ is reducible to $TrueSnt$.

5. Let $\mathcal{L} = (+, -, \cdot, 0, 1, <, =)$ be the language of ordered rings and

$$\mathcal{R} = (\mathbb{R}, +_{\mathbb{R}}, -_{\mathbb{R}}, \cdot_{\mathbb{R}}, 0_{\mathbb{R}}, 1_{\mathbb{R}}, <_{\mathbb{R}}, =_{\mathbb{R}})$$

the ordered field of real numbers.

(a) Give the definitions of an effective predicate A of reals (that is, $A \subset \mathbb{R}^n$) and the definition of an effective function $f : D \to \mathbb{R}$, $D \subset \mathbb{R}^n$.

(b) Show that the function $x \div y$ is effective.

(c) Outline the proof that if $p(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{R}$ and $a' = a_0, \ldots, a_n$, then there are $n + 1$ effective functions $\eta_1(a') < \ldots < \eta_k(a')$, and $k = k(a')$ such that $\eta_1(a') < \cdots < \eta_k(a')$ are all of the real roots of $p(x)$.