UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C381: Logic

COURSE CODE : **MATHC381**

UNIT VALUE : **0.50**

DATE : **13-MAY-03**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

03-C0926-3-70

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Consider an algebraic theory with operational type $\Omega := \{m_2\}$ and unique axiom $mxy = myx$. Show that the equation $mxx = x$ is not provable.

   Suppose that $\mathcal{S}$ is a consistent set of propositions (of the propositional calculus) which is complete in the following sense: for every proposition s either $s \in \mathcal{S}$ or $\neg s \in \mathcal{S}$. Show that $\mathcal{S}$ has a model.

   Using only the axioms of the propositional calculus, its rule of inference and the Deduction Theorem, show that

$$\vdash ((p \Rightarrow q) \Rightarrow (\neg q \Rightarrow \neg p)).$$

2. Sketch an argument showing that every consistent theory (in a countable language) can be enlarged to a complete consistent theory having witnesses.

   Consider a first order theory with operational type $\Omega = \{m_2\}$ and a unique axiom

$$(\exists x)(mxx = x).$$

   Explain why this theory does not have witnesses. Enlarge the theory so that the axiom has a witness.

3. Show that in ZFC for every set x there is a bijection of x onto an ordinal.

   Describe such a bijection for the set of all integers.

   Show that $2^m > m$ for every cardinal number m.

4. Write a program for the register machine computing the function

$$r(m, n) = \text{remainder when } m \text{ is divided by } n, \text{ provided that } n > 0$$

For simplification, the program may use, in addition to the basic instructions, also the instruction 'copy the content of R_k to R_l and go to S_j', denoted by $S_i \overset{R_k \to R_l}{\to} S_j$. Explain how your program works in the cases (a) $m = 10, n = 4$ and (b) $m = 7, n = 0$.

Use the previous part of the question to answer the following questions, stating the results that you use.

(a) Is $r(m, n)$ recursive?

(b) Is $r(m, n)$ **PA** definable?

5. Suppose that $\mathcal{T}$ is a mathematical theory containing the Peano Arithmetic. Explain the definition of a predicate of $\mathcal{T}$ which describes the meta-statement "$\mathcal{T}$ is consistent". (Any predicate of $\mathcal{T}$ used in the definition should be defined as well.)

State and prove Gödel's second incompleteness theorem. (The fixed point lemma and/or properties of the Gödel's sentence may be stated without a proof.)