UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. B.Sc.(Econ)M.Sci.

Mathematics M255: Linear Programming and Optimization

COURSE CODE	:	**MATHM255**
UNIT VALUE	:	**0.50**
DATE	:	**15-MAY-03**
TIME	:	**10.00**
TIME ALLOWED	:	**2 Hours**

03-C0959-3-140

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Let P be the linear program in equality form

$$\text{Maximise} \quad c^T.x$$

$$\text{subject to} \quad Ax = b \;,\; x \geqslant 0$$

where x and c are vectors in $\mathbf{R}^n$, A is an $m \times n$ matrix and b is a vector in $\mathbf{R}^m$.

Explain the term *basic feasible solution* (BFS) for P.

What does it mean for an array

$$\begin{array}{c|c} \tilde{A} & \tilde{b} \\ \hline \tilde{c}^T & -J \end{array}$$

to be a valid simplex tableau for P?

Show how to read off the BFS corresponding to such a tableau.

For this BFS, state (with justification) the value of the objective function. Show that if $\tilde{c} \leqslant 0$ then this BFS is optimal.

If $\tilde{c}$ has a positive entry, $\tilde{c}_j$, explain how to locate a pivot element, if it exists, describe the corresponding pivot move and confirm that after the move, the new vector $\tilde{b}$ will be non-negative.

2. Consider the following standard linear program, P.

$$\begin{array}{lrcrcrcr} \text{Maximise} & x & + & 2y & + & z & & \\ & & & & & & & \\ \text{subject to} & 3x & + & 2y & + & z & \leqslant & 8 \\ & x & + & 4y & + & z & \leqslant & 10 \\ & x & + & 2y & + & 3z & \leqslant & 8 \\ & x & , & y & , & z & \geqslant & 0 \end{array}$$

Write down the dual to P.

Use the simplex algorithm to find the optimal solution to P and give the value of the objective function at this optimal point. After each step, identify the basic feasible solution produced and check that it *is* feasible.

State the strong duality theorem for P and its dual.

Write down the optimal solution for the dual and confirm that it is feasible and that it yields the optimal value for the dual (as determined by strong duality).

3. Explain what is meant by a two-person zero-sum game with payoff matrix A.

What is meant by an *equilibrium*, a *saddle point*, a *pure strategy* and a *randomised strategy* for such a game.

Show that if a game has a saddle point then there is an equilibrium consisting of pure strategies.

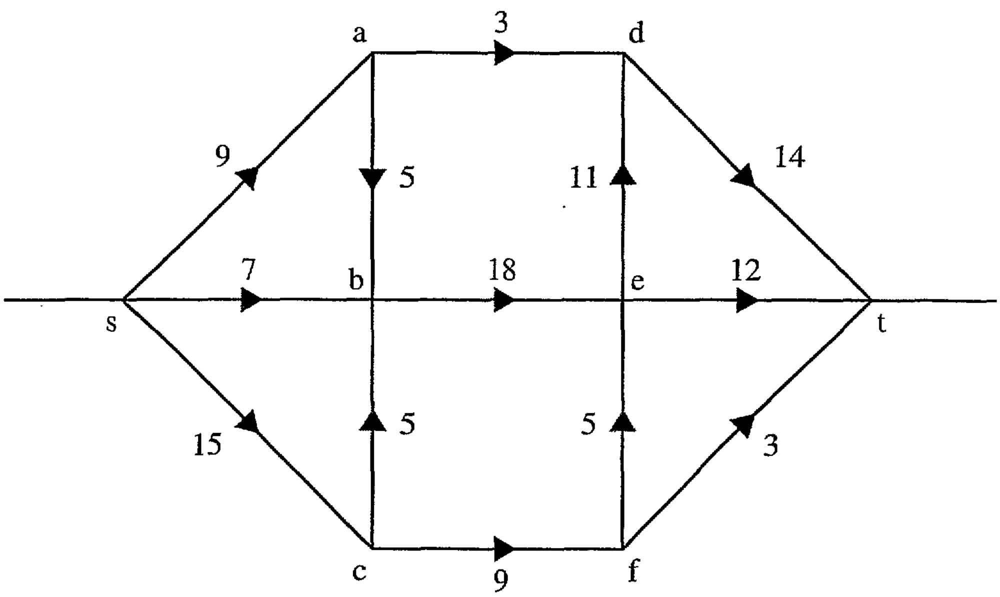
Explain, with justification, what condition should be satisfied by a pair of randomised strategies, if they are to constitute an equilibrium.

For each real value of a find the equilibrium for the game with payoff matrix

$$\begin{pmatrix} 2 & 3 \\ a & 1 \end{pmatrix}.$$

4. Prove that in a network with capacities, there is a cut whose value is equal to that of the maximum flow.

   Find a maximum flow and a minimum cut for the following network.



PLEASE TURN OVER

5. State and prove the Lagrangian sufficiency theorem.

Let f and g be the functions given by

$$\begin{aligned} f(x,y) &= 8x + y \\ g(x,y) &= x^4 + y^4 - 17. \end{aligned}$$

Show that f has a maximum and a minimum value on the set of points (x, y) in $\mathbf{R}^2$ which satisfy

$$g(x,y) = 0.$$

Use a Lagrange multiplier to find the maximum and minimum values and the points at which they occur.

6. Suppose that sources $S_1, S_2, \ldots, S_m$ can supply respectively $s_1, s_2, \ldots, s_m$ units of a certain product and n destinations $D_1, D_2, \ldots, D_n$ demand $d_1, d_2, \ldots, d_n$ units respectively, where

$$\sum s_i = \sum d_j.$$

Explain what is meant by a *transportation* from the sources to the destinations and show that at least one such transportation exists, with at most $m + n - 1$ non-zero entries.

Given a cost matrix C, whose ij^{th} entry c_{ij} is the cost per unit of transporting from source S_i to destination D_j, state what it means for a transportation to be optimal.

Find an optimal transportation for the array

	D_1	D_2	D_3	D_4	
S_1	4	5	3	3	4
S_2	6	6	3	7	7
S_3	2	8	4	3	7
	3	5	5	5	18