UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualification:-

M.Sci.

Mathematics C397: Lie groups, Lie algebras and supersymmetry

COURSE CODE	:	**MATHC397**
UNIT VALUE	:	**0.50**
DATE	:	**28-APR-05**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

05-C0945-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Let $\mathbf{g}$ be a finite dimensional Lie algebra over $\mathbb{F}$. Define the Killing form $\beta_{\mathbf{g}}$ of $\mathbf{g}$, and prove that for all $X, Y, Z \in \mathbf{g}$,

$$\beta_{\mathbf{g}}(X, [Y, Z])) = \beta_{\mathbf{g}}(Y, [Z, X])).$$

Show that if $\mathbf{a} \lhd \mathbf{g}$ then $\mathbf{a}^{\perp} \lhd \mathbf{g}$ where $\mathbf{a}^{\perp} = \{B \in \mathbf{g} : \forall A \in \mathbf{a}, \beta_{\mathbf{g}}(A, B) = 0\}$.

If $\mathbf{g}$ is simple deduce that *either* (i) $\beta_{\mathbf{g}}$ is nondegenerate *or* (ii) $\beta_{\mathbf{g}} \equiv 0$.

Prove that $\mathbf{sl}_3(\mathbb{F})$ is simple.

2. The following elements X, Y, Z form an $\mathbb{F}$-basis for the Lie algebra $\mathcal{O}(3, \mathbb{F})$;

$$X = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ; Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} ; Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Write down the relations which hold between X, Y, Z under Lie bracket, and hence compute the Killing form of $\mathcal{O}(3, \mathbb{F})$ in the coordinate system of $\{X, Y, Z\}$.

If $\varphi : \mathcal{O}(3, \mathbb{F}) \to \mathbf{gl}_{\mathbb{F}}(V)$ is a representation of $\mathcal{O}(3, \mathbb{F})$, write down the Casimir operator C_φ of φ and show by direct computation that

$$C_\varphi \varphi(X) = \varphi(X) C_\varphi.$$

Deduce that $C_\varphi \varphi(\xi) = \varphi(\xi) C_\varphi$ for all $\xi \in \mathcal{O}(3, \mathbb{F})$.

3. For any nonnegative half integer j define the spin j representation (j) of $\mathbf{sl}_2(\mathbb{F})$, and show how to interpret the adjoint representation of $\mathbf{sl}_2(\mathbb{F})$ as a spin representation.

Prove that

$$(j) \otimes (\tfrac{1}{2}) \;\cong\; (j - \tfrac{1}{2}) \oplus (j + \tfrac{1}{2}).$$

Hence derive a corresponding decomposition formula for $(j) \otimes (1)$, stating clearly any uniqueness statement about $\mathbf{sl}_2(\mathbb{F})$ that you use.

4. Let $\mathbf{g}$ be a finite dimensional nondegenerate Lie algebra over $\bar{\mathbb{F}}$ and let (V, φ) be a finite dimensional representation of $\mathbf{g}$.

   Explain what is meant by a 1-cocycle of $\mathbf{g}$ with values in (V, φ), and explain also what is meant by saying that such a 1-cocycle is trivial.

   Furthermore, if (V, φ) is simple and the Casimir operator C_φ of (V, φ) is nonzero, show that any 1-cocycle of $\mathbf{g}$ with values in (V, φ) is trivial.

   Explain what is meant by the *Whitehead property* for $\mathbf{g}$.

   Let $\mathbf{g}$ have the Whitehead property; given a short exact sequence of $\mathbf{g}$-representations over $\bar{\mathbb{F}}$
   $$0 \to (U_1, \alpha_1) \to (W, \psi) \to (U_2, \alpha_2) \to 0$$
   show that $(W, \psi) \cong (U_1, \alpha_1) \oplus (U_2, \alpha_2)$.

5. Let $\mathbf{g}$ be a Lie algebra of finite dimension over $\mathbb{C}$. Explain the distinction between the following sorts of representation of $\mathbf{g}$;

   (i) $(\mathbb{C}, \mathbb{C})$ -representation ; (ii) $(\mathbb{R}, \mathbb{C})$ -representation ; (iii) $(\mathbb{R}, \mathbb{R})$ -representation.

   Also describe a 1-1 correspondence
   $$\{(\mathbb{R}, \mathbb{C})\text{−representations of } \mathbf{g}\} \Leftrightarrow \{(\mathbb{C}, \mathbb{C})\text{−representations of } \mathbf{g} \oplus \bar{\mathbf{g}}\}.$$

   Describe by means of bispinors the classification of

   (i) simple $(\mathbb{R}, \mathbb{C})$-representations of $\mathbf{sl}_2(\mathbb{C})$, and

   (ii) simple $(\mathbb{R}, \mathbb{R})$-representations of $\mathbf{sl}_2(\mathbb{C})$.

   Viewed as $(\mathbb{R}, \mathbb{R})$-representations of $\mathbf{sl}_2(\mathbb{C})$ the *Majorana* representation $\mathcal{M}$ and *vector* representation $\mathcal{V}$ are described by
   $$\mathcal{M} = (\frac{1}{2}, 0) \oplus (0, \frac{1}{2}) \quad ; \quad \mathcal{V} = (\frac{1}{2}, \frac{1}{2}).$$

   By using the bispinor Clebsch-Gordan Theorem calculate the bispinor decompositions of both $\mathcal{M} \otimes \mathcal{V}$ and $\mathcal{V} \otimes \mathcal{V}$.