UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualification:-

M.Sci.

Mathematics M312: Large Deviations in Probability and Geometry

COURSE CODE	:	**MATHM312**
UNIT VALUE	:	**0.50**
DATE	:	**29-MAY-03**
TIME	:	**10.00**
TIME ALLOWED	:	**2 Hours**

03-C0962-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. State the Brunn-Minkowski inequality for compact sets in $\mathbb{R}^n$.

   Prove that if $f, g, m : \mathbb{R} \to [0, \infty)$ are measurable, $\lambda \in (0,1)$ and for all $x, y \in \mathbb{R}$

   $$m((1-\lambda)x + \lambda y) \geqslant f(x)^{1-\lambda} g(y)^{\lambda},$$

   then

   $$\int m \geqslant \left(\int f\right)^{1-\lambda} \left(\int g\right)^{\lambda}.$$

   Explain briefly how this inequality can be used to prove the Brunn-Minkowski inequality.

2. State a form of the isoperimetric inequality on the unit sphere in $\mathbb{R}^n$ equipped with rotation invariant probability σ.

   State Dvoretzky's Theorem on the existence of almost Euclidean subspaces of normed spaces. Explain, without including technical details, how it can be proved using an isoperimetric inequality of the above type. You may assume that if K is the unit ball of a norm $\|.\|$ on $\mathbb{R}^n$ and that the standard Euclidean ball is the ellipsoid of maximal volume inside K, then

   $$\int_{S^{n-1}} \|\theta\| \, d\sigma(\theta) \geqslant c\frac{\sqrt{\log n}}{\sqrt{n}}$$

   for some constant $c > 0$ independent of n and K.

3. The logarithmic Sobolev inequality of Gross, states that for sufficiently regular functions $f : \mathbb{R} \to (0, \infty)$

$$\int f \log f - \left(\int f\right) \log\left(\int f\right) \leqslant \frac{1}{2}\int \frac{f'^2}{f} + C\int f.$$

The time-dependent density $F : (t, x) \mapsto F(t, x)$ on $(0, \infty) \times \mathbb{R}$ evolves according to the heat equation

$$\partial_t F = \partial_{xx} F.$$

Let $p = p(t) = 1 + e^{2t}$ so that $p'(t) = 2(p - 1)$. Assuming sufficient regularity of F, prove that

$$\frac{d}{dt}\int F^{p(t)}\, dx = \frac{2(p-1)}{p}\int F^p \log F^p - p(p-1)\int F^{p-2}F'^2.$$

Deduce that

$$\frac{d}{dt}\|F\|_p \leqslant \frac{2C(p-1)}{p^2}\|F\|_p.$$

Explain briefly what this tells you about the behaviour of the time evolutes of the density $x \mapsto F(t, x)$.

4. Given two probability measures μ and ν on $\mathbb{R}^n$ explain what is meant by a *transportation* map carrying μ to ν.

   What special characteristics are possessed by the Brenier transportation (if it exists)?

   Now suppose that μ is absolutely continuous with respect to Lebesgue measure and ν has compact support. Show that the Brenier transportation does indeed exist. (You may assume standard properties of convex functions on $\mathbb{R}^n$: for example that they are differentiable almost everywhere.)

5. Prove the inequality of Brascamp and Lieb which states that if (u_i) is a sequence of unit vectors in $\mathbb{R}^n$ and (c_i) is a sequence of positive numbers satisfying the John condition

$$\sum c_i u_i \otimes u_i = I_n,$$

and $f_i : \mathbb{R} \to [0, \infty)$ are measurable then

$$\int_{\mathbb{R}^n} \prod_1^m f_i(\langle x, u_i \rangle)^{c_i} dx \leqslant \prod_1^m \left(\int f_i \right)^{c_i}.$$

(You may assume that for any sequence (α_i) of positive numbers

$$\det \left(\sum c_i \alpha_i u_i \otimes u_i \right) \geqslant \prod \alpha_i^{c_i}.)$$

Show that there is equality in the Brascamp-Lieb inequality if the f_i are identical Gaussian densities.