UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics 2503: Introductory Mathematical Biology

COURSE CODE : **MATH2503**

UNIT VALUE : **0.50**

DATE : **22-MAY-06**

TIME : **10.00**

TIME ALLOWED : **2 Hours**

06-C0989-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Consider the chemical reaction scheme in which X converts to Y at rate k_1, Y converts back to X at rate k_{-1} and also forward to the product P:

$$X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} Y \overset{k_2}{\rightarrow} P.$$

Let $x(t), y(t)$ be the concentrations of X, Y at time t.

(a) Write down the differential equations for the evolution of x and y and show that they can be written in the form $\frac{d\mathbf{x}}{dt} = M\mathbf{x}$ where $\mathbf{x} = (x, y)^T$ and M is a matrix which you should find.

(b) Now suppose that $k_1 = 2, k_{-1} = 1$ and $k_2 = 1$. Find an explicit expression for the evolution of the concentrations of X, Y given that initially X has concentration x_0 and Y has concentration y_0.

(c) Sketch the phase plane (i.e. solution curves in $(x, y)-$space) to indicate what happens to solutions $(x(t), y(t))$ as $t \to \infty$ for different initial conditions.

2. An ion channel model has 2 gates, one at each end, that may be either open or closed so that it has 3 states: state 1, the open-open state; state 2, the open-closed state; state 3, the closed-open state. The closed-closed state is to be ignored. The probability per unit time of a channel moving from state i to state j is denoted by k_{ij} for $i, j = 1, 2, 3$. There are no transitions from a state i to state i for $i = 1, 2, 3$. Let $\mathbf{p}(t) = (p_1(t), p_2(t), p_3(t))$ be the probability distribution of the channel states at time t.

(a) Write down a Markov chain model for the evolution of $\mathbf{p}(t)$.

(b) Suppose that $k_{12} = k_{13} = k_{21} = k_{23} = k_{31} = k_{32} = 1$. State what property of the chain ensures that it converges to a unique steady state and find the steady state.

(c) Find the long term probability distribution of the channel states if $k_{13} = k_{21} = k_{23} = k_{31} = 1$, but $k_{12} = k_{32} = 0$.

3. Suppose M is an invertible real $n \times n$ matrix. Consider the initial value problem

$$\frac{d\mathbf{x}}{dt} = M\mathbf{x} + \mathbf{a}, \ t \geqslant 0, \ \mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^n. \tag{1}$$

(a) Find an explicit expression for the solution $\mathbf{x}(t)$ of (1).

(b) Suppose that the eigenvalues of M are real. Briefly discuss how the long term behaviour of the solution $\mathbf{x}(t)$ depends upon the eigenvalues of M.

A drug accumulates in the blood at constant rate γ ($\mathrm{s}^{-1}\mathrm{mol}^{-1}$). From the blood it transfers to the tissue at rate α (s^{-1}) and then from the tissue it transfers to the liver at rate β (s^{-1}) where it is removed at rate δ (s^{-1}).

(c) Denoting the concentration of the drug in the blood, tissue and liver by $B(t)$, $T(t)$, $L(t)$ respectively, write down a differential equation model for the drug transfer.

(d) Find the limiting concentrations $B(\infty)$, $T(\infty)$, $L(\infty)$. [You are not required to give expressions for $B(t)$, $T(t)$, $L(t)$].

4. Consider the one-dimensional passive diffusion of a chemical X through a membrane of thickness h spanning from $x = 0$ to $x = h$, and with diffusive constant D. For $x \leqslant 0$ the concentration of X is maintained at the constant value C_0 and for $x \geqslant h$ it is maintained at the constant value C_1.

(a) Write down the partial differential equation for the evolution of the concentration $C(x,t)$ of X in the interval $x \in [0,h]$, together with the model boundary conditions.

(b) Suppose that the concentration of X within the membrane is in steady state. Find the profile of the concentration of X within the membrane.

Now suppose that there are two membranes, one spanning the interval $[0,h)$ with diffusive constant D_1 and the second spanning the interval $[h,2h]$ with diffusive constant D_2. The concentration of X for $x \leqslant 0$ is maintained at the constant value C_0 and the concentration of X for $x \geqslant 2h$ is maintained at the constant value C_1.

(c) Find the stationary profile of the concentration of X within the two membranes.

5. Consider the system of 3 blood vessels in which vessel 1 bifurcates into vessel 2 and vessel 3, as shown in figure 1. For $i = 1, 2, 3$ let Q_i denote the flux in vessel i, and P_i the fixed pressure at the end of vessel i. Vessel 1 has length 2 units and vessels 2

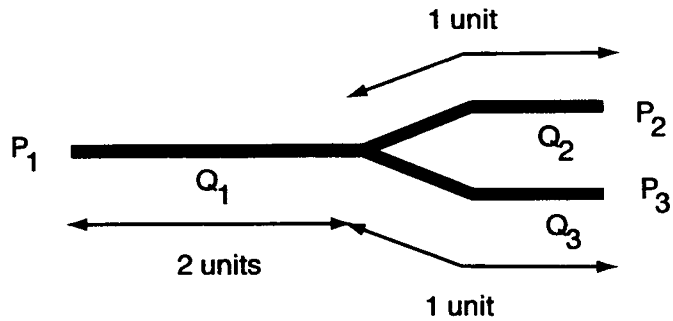

Figure 1:

and 3 have unit length. Ohm's law for each vessel is $\Delta P = \alpha L Q$ where ΔP is the pressure difference across the vessel, Q is the flux, L is the vessel length, and $\alpha > 0$ is a constant (same for all vessels).

(a) Find the fluxes Q_1, Q_2, Q_3 in terms of the pressures P_1, P_2, P_3 and α.

Now suppose that Ohm's law for vessel 1 is replaced by the new law $\Delta P = \alpha L|Q|Q$ but remains $\Delta P = \alpha L Q$ for vessels 2 and 3.

(b) Show that the new flux Q_1 satisfies

$$|Q_1|Q_1 + \frac{Q_1}{4} + \frac{1}{4\alpha}(P_3 + P_2 - 2P_1) = 0.$$

(c) For fixed pressures $P_1 > P_2$ and $P_1 > P_3$ how many possible fluxes Q_1 are there? Justify your answer.

END OF PAPER