UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualification:-

M.Sc.

Geophysical Fluid Dynamics

COURSE CODE	:	**MATHG304**
DATE	:	**11-MAY-06**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

06-N0124-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

The fluid is incompressible and, except in Question 5, has constant density ρ. Gravitational acceleration is denoted by g throughout. The Coriolis parameter is denoted by f and is taken to be constant throughout.

The shallow water equations can be written

$$u_t + uu_x + vu_y - fv = -g\eta_x,$$

$$v_t + uv_x + vv_y + fu = -g\eta_y,$$

$$H_t + (uH)_x + (vH)_y = 0,$$

where H is the total depth and $\eta(x, y, t)$ the free surface displacement.

1. (a) Use the three-dimensional continuity equation, the free surface boundary condition and the fact that horizontal velocities are independent of depth to show that the vertical component of velocity associated with the shallow water equations can be written

$$w = -(u_x + v_y)(z - \eta) + \eta_t + u\eta_x + v\eta_y.$$

   (b) Hence, or otherwise, show that the shallow water equations imply that the fractional depth of a particle in a fluid column remains constant throughout any motion.

   (c) Similarly, or otherwise, show that the shallow water equations imply that the potential vorticity

$$q = (v_x - u_y + f)/H$$

   of a particle remains constant throughout any motion.

   (d) Hence, or otherwise, describe briefly the mechanism whereby an infinitesimal sinusoidal disturbance to a material line of particles initially lying along a straight isobath propagates along the isobath.

2. (a) From the linearised shallow water equations for a fluid of constant undisturbed depth H_0, so $H = H_0 + \eta$, derive the linearised equation for the conservation of potential vorticity, i.e.

$$\frac{\partial}{\partial t}\left(\zeta - \frac{f\eta}{H_0}\right) = 0,$$

where $\zeta = v_x - u_y$ and f is constant.

(b) Using this result, or otherwise, find the final steady-state flow when the free surface displacement $\eta(x,t)$ evolves from an initial state of rest with

$$\zeta(x,0) = 0, \qquad \eta(x,0) = -\eta_0 \mathrm{sgn} x.$$

(c) Show that the *increase* in potential energy (per unit width in the y-direction) of a fluid strip of length δx when the surface moves from η_1 to η_2 is

$$\frac{1}{2}\rho g(\eta_2^2 - \eta_1^2)\delta x.$$

Hence show that during the adjustment to a steady state in (b) the total potential energy per unit width released is $\frac{3}{2}\rho g \eta_0^2 a$ where $a = (gH_0)^{1/2}/f$.

(d) Show that the kinetic energy per unit width of a strip of length δx is

$$\frac{1}{2}\rho H_0 g^2 f^{-2}(\eta_x)^2 \delta x$$

and hence that the total increase in kinetic energy during the adjustment in (b) is $\frac{1}{2}\rho g \eta_0^2 a$.

(e) How much energy is "missing" and where has it gone?

3. The quasigeostrophic potential vorticity equation can be written

$$(\partial_t + \psi_x \partial_y - \psi_y \partial_x)(\nabla^2 \psi - F\psi + \eta_B) = 0,$$

where F is a number measuring surface deformation, $\eta_B(x,y)$ gives the shape of the lower boundary, and ψ is a streamfunction for the motion.

(a) Show that when the flow field is unbounded and the bottom slopes uniformly so that $\eta_B = \beta y$ this equation admits *finite amplitude* wave motion of the form

$$\psi = A\cos(kx + ly - \sigma t),$$

where A, k, l and σ are constants. Derive the dispersion relation for these waves.

(b) Derive and sketch a geometric relationship in wavenumber space between the group and phase velocities of waves of fixed frequency σ (and infinitesimal amplitude).

(c) Use this relationship to describe the oblique reflection of a Rossby wave in the quarter plane $x > 0$, $y > 0$ which first reflects off the solid barrier $y = 0$ and travels onwards to reflect off the solid barrier $x = 0$.

4. A viscous fluid, occupying the region $z < 0$, is rotating at uniform angular speed Ω about the vertical Oz axis. A *steady* flow is driven by a constant stress $\tau\hat{\boldsymbol{x}}$ (where $\hat{\boldsymbol{x}}$ is a horizontal unit vector in the rotating frame) applied at the surface $z = 0$.

The momentum and continuity equations for the flow can be written

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + 2\Omega\hat{\boldsymbol{z}} \wedge \boldsymbol{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{u},$$

$$\nabla \cdot \boldsymbol{u} = 0,$$

where $\boldsymbol{u}$ is the velocity relative to the rotating frame, $\hat{\boldsymbol{z}}$ is a vertical unit vector, p is the pressure, ρ is the constant fluid density and ν is the constant kinematic viscosity of the fluid.

(a) Solve these equations to obtain the velocity components of the flow relative to the rotating axes.

(b) Find the magnitude of the induced surface velocity and its direction relative to the applied stress.

(c) Discuss the variation of the velocity with depth.

5. The governing equations for a Boussinesq incompressible fluid can be written, for Oz vertical, as

$$\begin{aligned} \boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} &= -\frac{1}{\rho}\nabla p + \sigma\hat{\boldsymbol{z}}, \\ \sigma_t + (\boldsymbol{u} \cdot \nabla)\sigma + N^2 w &= 0, \\ \nabla \cdot \boldsymbol{u} &= 0, \end{aligned}$$

where $\sigma = g(\delta\rho/\rho)$ is the buoyancy acceleration and N^2 is the buoyancy frequency.

(a) Derive the internal wave equation for the pressure p, governing *small* oscillations in a fluid when N^2 is constant.

(b) Derive a geometric relation between the group and phase velocities of the waves.

(c) Consider a vertically semi-infinite stratified fluid with constant N^2 above a sinusoidal boundary

$$z = \epsilon \sin\{k(x - Ut)\}$$

with wavenumber k, height $\epsilon \ll 1$, and travelling in the positive x-direction at speed U. Discuss the form of the motion for $N < kU$ and $N > kU$, obtaining the slope of the phase lines and the direction of energy propagation of any waves excited.