UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C344: Geophysical Fluid Dynamics

COURSE CODE : **MATHC344**

UNIT VALUE : **0.50**

DATE : **24-MAY-05**

TIME : **10.00**

TIME ALLOWED : **2 Hours**

05-C0930-3-50

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

The fluid is incompressible and has constant density ρ. Gravitational acceleration is denoted by g throughout. The Coriolis parameter is denoted by f throughout.

1. The shallow water equations can be written

$$\begin{aligned} u_t + uu_x + vu_y - fv &= -g\eta_x, \\ v_t + uv_x + vv_y + fu &= -g\eta_y, \\ H_t + (uH)_x + (vH)_y &= 0, \end{aligned}$$

where $H(x, y, t)$ is the total depth and $\eta(x, y, t)$ the free surface displacement.

(a) Show that the linearised shallow water momentum equations can be written

$$(\partial_{tt} + f^2)u = -g(\eta_{xt} + f\eta_y),$$

$$(\partial_{tt} + f^2)v = -g(\eta_{yt} - f\eta_x).$$

(b) Consider the free modes in a channel of constant width L, bounded by rigid walls at $y = 0$ and $y = L$, when the channel floor slopes linearly in the y-direction so that the undisturbed depth is given by

$$H_0(x, y) = D(1 + sy/L),$$

for constants D and s, with $s \ll 1$.
Discuss the departures of these modes from their flat-bottomed ($s = 0$) form and plot the leading order (in s) dispersion relation for the modes.
You may assume without proof that the governing equation for η of the form

$$\eta(x, y, t) = \mathcal{R}\{\bar{\eta}(y)\exp[\mathrm{i}(kx - \sigma t)]\},$$

with k and σ constants, is

$$\frac{\mathrm{d}^2\bar{\eta}}{\mathrm{d}y^2} + [(\sigma^2 - f^2)/c^2 - k^2 - fsk/(L\sigma)]\bar{\eta} = 0,$$

where $c^2 = gD$.
You may ignore any waves whose frequency in the flat-bottomed case ($s = 0$) is given by $\sigma = \pm f$ or $\sigma = \pm ck$.

2. The quasigeostrophic potential vorticity equation can be written

$$(\partial_t + \psi_x\partial_y - \psi_y\partial_x)(\nabla^2\psi - F\psi + \eta_B) = 0,$$

where F is an order unity number, measuring free-surface effects, $\eta_B(x, y)$ gives the shape of the lower boundary, and ψ is a streamfunction for the motion.

(a) Solve this equation for the steady motion in the half-plane $x > 0$ when the flow is bounded by an impermeable wall at $x = 0$, the flow far from the wall is directed towards the wall and has uniform speed unity, and the lower boundary slopes uniformly such that $\eta_B = \beta y$ with $\beta > 0$.

(b) Discuss the form of the solution for large β, relating expressions for the mass flux across planes $x =$*constant* to the flux across planes $y =$*constant.* Comment briefly on the solution if β is negative.

3. (a) Show that the quasigeostrophic potential vorticity equation (given in Question 2) admits *finite amplitude* wave motion of the form

$$\psi = \text{A}\cos(kx + ly - \sigma t) \tag{1}$$

when the flow field is unbounded and the bottom slopes uniformly such that $\eta_B = \beta y$.

(b) By linearising the quasigeostrophic potential vorticity equation and multiplying by ψ, derive the energy conservation law

$$E_t + \nabla \cdot \mathbf{S} = 0,$$

where $E = \frac{1}{2}|\nabla\psi|^2 + \frac{1}{2}F\psi^2$ and $\mathbf{S} = -\psi\nabla\psi_t - \frac{1}{2}\hat{\imath}\beta\psi^2$.

(c) By averaging over a period (denoted by $< \cdot >$) for a wave of form (1) show that

$$< \mathbf{S} >= \mathbf{c}_g < E >,$$

where $\mathbf{c}_g = \nabla_k\sigma$ is the group velocity. Hence show that

$$< E >_t + (\mathbf{c}_g \cdot \nabla) < E >= 0,$$

and deduce that the energy of the motion travels with the group velocity.

(d) Using the expression derived above for $< \mathbf{S} >$, or otherwise, derive a diagram showing the geometric relationship between the group and phase velocities of waves of fixed frequency σ.

4. By considering the Navier-Stokes equations for a layer of almost-inviscid fluid (of kinematic viscosity ν) rotating rapidly (with angular speed 2Ω) about a vertical axis Oz, *derive* the Ekman compatibility condition,

$$w = \frac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}(v_x - u_y),$$

imposed on an interior flow by the Ekman layer on an impermeable, co-rotating horizontal lower boundary. Here (x, y, z) are Cartesian co-ordinates and the corresponding components of the interior velocity are (u, v, w).

5. In a region sufficiently close to the equator that f can be approximated by βy, the constant-depth linearised shallow-water equations become (suitably non-dimensionalised)

$$u_t - \frac{1}{2}yv = -\eta_x,$$

$$v_t + \frac{1}{2}yu = -\eta_y,$$

$$\eta_t + u_x + v_y = 0.$$

Eliminating u and η gives the governing equation

$$v_{ttt} + \frac{1}{4}y^2 v_t - (v_{xx} + v_{yy})_t - \frac{1}{2}v_x = 0.$$

(a) Using the equation for v alone, find the dispersion relation for waves in an unbounded domain.

(b) By considering the original system, show that $v = 0$ gives a non-trivial solution.

(c) Sketch the dispersion relation, showing intersections with axes, asymptotic behaviour and turning points.

[You may use without proof the result that the solutions of the eigenvalue problem

$$D_{yy} + (\lambda - \frac{1}{4}y^2)D = 0, \quad \text{with} \quad D \to 0 \quad \text{as} \quad |y| \to \infty,$$

are the parabolic cylinder functions $D_n(y)$, with $\lambda = n + \frac{1}{2}$ for $n = 0, 1, 2, ...$]