# UNIVERSITY COLLEGE LONDON

University of London

## EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

*B.Sc.* *M.Sci.*

**Mathematics C344: Geophysical Fluid Dynamics**

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| COURSE CODE | : | **MATHC344** |
| UNIT VALUE | : | **0.50** |
| DATE | : | **13-MAY-04** |
| TIME | : | **14.30** |
| TIME ALLOWED | : | **2 Hours** |

04-C0925-3-40


**TURN OVER**

*All questions may be attempted but only marks obtained on the best* **four** *solutions will count.*
*The use of an electronic calculator is* **not** *permitted in this examination.*

The fluid is incompressible and has constant density $\rho$. Gravitational acceleration is denoted by $g$ throughout. The Coriolis parameter may be taken as constant and is denoted by $f$ throughout.

The shallow water equations can be written

$$u_t + uu_x + vu_y - fv = -g\eta_x,$$

$$v_t + uv_x + vv_y + fu = -g\eta_y,$$

$$H_t + (uH)_x + (vH)_y = 0,$$

where $H$ is the total depth and $\eta(x, y, t)$ the free surface displacement.

1. (a) Use the three-dimensional continuity equation, the free surface boundary condition and the fact that horizontal velocities are independent of depth to show that the vertical component of velocity associated with the shallow water equations can be written

$$w = -(u_x + v_y)(z - \eta) + \eta_t + u\eta_x + v\eta_y.$$

   (b) Hence, or otherwise, show that the shallow water equations imply that the fractional depth of a particle in a fluid column remains constant throughout any motion.

   (c) Similarly, or otherwise, show that the shallow water equations imply that the potential vorticity

$$q = (v_x - u_y + f)/H$$

   of a particle remains constant throughout any motion.

   (d) Hence, or otherwise, describe briefly the mechanism whereby an infinitesimal sinusoidal disturbance to a material line of particles initially lying along a straight isobath propagates along the isobath.

2. (a) Show that for a fluid of constant undisturbed depth $H_0$ (so $H = H_0 + \eta$) the linearised shallow water momentum equations become

$$(\partial_{tt} + f^2)u = -g(\eta_{xt} + f\eta_y),$$

$$(\partial_{tt} + f^2)v = -g(\eta_{yt} - f\eta_x).$$

Hence show that $\eta$ satisfies

$$[(\partial_{tt} + f^2)\eta - c^2(\eta_{xx} + \eta_{yy})]_t = 0,$$

where $c^2 = gH_0$.

(b) Find and plot the dispersion relation for the free modes in a channel of constant undisturbed depth $H_0$ and constant width $L$, bounded by rigid walls at $y = 0$ and $y = L$. You may ignore any waves of frequency $\sigma$ equal to $f$.

3. The quasigeostrophic potential vorticity equation can be written

$$(\partial_t + \psi_x\partial_y - \psi_y\partial_x)(\nabla^2\psi - F\psi + \eta_B) = 0,$$

where $F$ is a number measuring surface deformation, $\eta_B(x, y)$ gives the shape of the lower boundary, and $\psi$ is a streamfunction for the motion.

(a) Show that when the flow field is unbounded and the bottom slopes uniformly so that $\eta_B = \beta y$ this equation admits *finite amplitude* wave motion of the form

$$\psi = A\cos(kx + ly - \sigma t),$$

where $A$, $k$, $l$ and $\sigma$ are constants. Derive the dispersion relation for these waves.

(b) Derive and sketch a geometric relationship in wavenumber space between the group and phase velocities of waves of fixed frequency $\sigma$ (and infinitesimal amplitude).

(c) Use this relationship to describe the reflection from a solid barrier at $y = x$ of a Rossby wave incident from $y > x$.

4. A viscous fluid, occupying the region $z < 0$, is rotating at uniform angular speed $\Omega$ about the vertical $Oz$ axis. A *steady* flow is driven by a constant stress $\tau\hat{\boldsymbol{x}}$ (where $\hat{\boldsymbol{x}}$ is a horizontal unit vector in the rotating frame) applied at the surface $z = 0$.

The momentum and continuity equations for the flow can be written

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + 2\Omega\hat{\boldsymbol{z}} \wedge \boldsymbol{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{u},$$

$$\nabla \cdot \boldsymbol{u} = 0,$$

where $\boldsymbol{u}$ is the velocity relative to the rotating frame, $\hat{\boldsymbol{z}}$ is a vertical unit vector, $p$ is the pressure, $\rho$ is the constant fluid density and $\nu$ is the constant kinematic viscosity of the fluid.

(a) Solve these equations to obtain the velocity components of the flow relative to the rotating axes.

(b) Find the magnitude of the induced surface velocity and its direction relative to the applied stress.

(c) Discuss the variation of the velocity with depth.

5. The linearised barotropic vorticity equation for a homogeneous ocean of depth $D$ and density $\rho$, forced by a surface wind stress $\boldsymbol{\tau}$ can be written

$$\nabla\psi_t + \beta\psi_x = -r\nabla^2\psi + (1/\rho D)\hat{\boldsymbol{z}} \cdot (\nabla \wedge \boldsymbol{\tau}),$$

where $\psi$ is a streamfunction for the depth-averaged motion and $\hat{\boldsymbol{z}}$ is a vertical unit vector.

(a) Briefly identify each term in the equation.

(b) Consider the *steady* flow in the rectangular basin $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$ forced by the wind stress

$$\boldsymbol{\tau} = -\tau_0 \cos(\pi y/b)\hat{\boldsymbol{x}}.$$

(i) By non-dimensionalising the equations and considering the limit

$$\epsilon_S = r/\beta a \ll 1,$$

show that a Sverdrup balance holds in the bulk of the ocean. Show that, at best, a Sverdrup solution satisfies the impermeability condition at either $x = 0$ or $x = a$ but not at both simultaneously.

(ii) By considering the dynamics of a layer of thickness $\epsilon_S$, obtain a composite form for the streamfunction in the basin and sketch the resulting flow pattern.