UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C344: Geophysical Fluid Dynamics

COURSE CODE	:	**MATHC344**
UNIT VALUE	:	**0.50**
DATE	:	**21-MAY-03**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

03-C0919-3-40

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

The fluid is incompressible and, except in Question 5, has constant density ρ. Gravitational acceleration is denoted throughout by g. The Coriolis parameter is denoted by f and is taken to be constant throughout.

The shallow water equations can be written

$$u_t + uu_x + vu_y - fv = -g\eta_x,$$

$$v_t + uv_x + vv_y + fu = -g\eta_y,$$

$$H_t + (uH)_x + (vH)_y = 0,$$

where H is the total depth and $\eta(x, y, t)$ is the free surface displacement.

1. (a) Show that, when f is constant, the linearised momentum equations can be written

$$(\partial_{tt} + f^2)\mathbf{u} = -g(\nabla\eta_t - f\mathbf{z} \wedge \nabla\eta),$$

where $\mathbf{z}$ is a unit vertical vector and ∇ is the horizontal gradient operator.

(b) Hence show that for a fluid of constant undisturbed depth H_0 (so $H = H_0 + \eta$) the displacement η satisfies

$$[(\partial_{tt} + f^2)\eta - c^2(\eta_{xx} + \eta_{yy})]_t = 0,$$

where $c^2 = gH_0$.

(c) Derive an equation whose roots give the frequencies of the normal modes of oscillation of the free surface of a rotating shallow cylindrical basin of radius L.

[You are given that the solutions of

$$r^2R'' + rR' + (r^2 - m^2)R = 0,$$

finite at $r = 0$, are the Bessel functions, $J_m(r)$.]

2. (a) From the linearised shallow water equations for a fluid of constant undisturbed depth H_0, so $H = H_0 + \eta$, derive the linearised equation for the conservation of potential vorticity, i.e.

$$\left(\zeta - \frac{f\eta}{H_0}\right)_t = 0,$$

where $\zeta = v_x - u_y$ and f is constant.

(b) Using this result, or otherwise, find the final steady-state flow when the free surface displacement $\eta(x,t)$ evolves from an initial state of rest with

$$\zeta(x,0) = 0, \qquad \eta(x,0) = -\eta_0 \text{sgn} x.$$

(c) Show that the *increase* in potential energy (per unit width in the y-direction) of a fluid strip of length δx when the surface moves from η_1 to η_2 is

$$\frac{1}{2}\rho g(\eta_2^2 - \eta_1^2)\delta x.$$

Hence show that during the adjustment to a steady state in (b) the total potential energy per unit width released is $\frac{3}{2}\rho g \eta_0^2 a$ where $a = (gH_0)^{1/2}/f$.

(d) Show that the kinetic energy per unit width of a strip of length δx is

$$\frac{1}{2}\rho H_0 g^2 f^{-2}(\eta_x)^2 \delta x$$

and hence that the total increase in kinetic energy during the adjustment in (b) is $\frac{1}{2}\rho g \eta_0^2 a$.

(e) How much energy is "missing" and where has it gone?

3. The quasigeostrophic potential vorticity equation can be written

$$(\partial_t + \psi_x \partial_y - \psi_y \partial_x)(\nabla^2 \psi - F\psi + \eta_B) = 0,$$

where F is a number measuring surface deformation, $\eta_B(x, y)$ gives the shape of the lower boundary, and ψ is a streamfunction for the motion.

(a) Show that when the flow field is unbounded and the bottom slopes uniformly so that $\eta_B = \beta y$ this equation admits *finite amplitude* wave motion of the form

$$\psi = A \cos(kx + ly - \sigma t), \qquad (1)$$

where A, k, l and σ are constants. Derive the dispersion relation for these waves.

(b) By linearising the quasigeostrophic potential vorticity equation and multiplying by ψ, derive the energy conservation law

$$E_t + \nabla \cdot \mathbf{S} = 0,$$

where $E = \frac{1}{2}|\nabla\psi|^2 + \frac{1}{2}F\psi^2$ and $\mathbf{S} = -\psi\nabla\psi_t - \frac{1}{2}\hat{\imath}\beta\psi^2$.

(c) By averaging over a period (denoted by $< \cdot >$) for a wave of form (1) show that

$$< \mathbf{S} >= \mathbf{c}_g < E >,$$

where $\mathbf{c}_g = \nabla_k \sigma$ is the group velocity. Hence show that

$$< E >_t + (\mathbf{c}_g \cdot \nabla) < E >= 0,$$

and deduce that the energy of the motion travels with the group velocity.

(d) Using the expression derived above for $< \mathbf{S} >$, or otherwise, derive a diagram showing the geometric relationship between the group and phase velocities of waves of fixed frequency σ.

4. A *viscous* fluid occupies the region $z > 0$ above a horizontal rigid plane $z = 0$. The plane is rotating with uniform angular speed Ω about the vertical axis Oz and the Cartesian axes $Oxyz$ rotate with the plane. Far above the plane ($z \gg 1$) the fluid velocity relative to these axes becomes horizontal with uniform speed U in the Ox direction.

The momentum and continuity equations for the flow can be written

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + 2\Omega\hat{\boldsymbol{k}} \wedge \boldsymbol{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{u},$$

$$\nabla \cdot \boldsymbol{u} = 0,$$

where $\boldsymbol{u}$ is the velocity relative to the rotating frame, $\hat{\boldsymbol{k}}$ is a vertical unit vector, p is the pressure, ρ is the constant fluid density and ν is the constant kinematic viscosity of the fluid.

(a) Solve these equations to obtain the velocity components of the steady flow relative to the rotating axes.

(b) Describe the variation with height of this velocity field.

5. The governing equations for a Boussinesq incompressible fluid can be written, for Oz vertical, as

$$\begin{aligned} \boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} &= -\frac{1}{\rho}\nabla p + \sigma\hat{\boldsymbol{z}}, \\ \sigma_t + (\boldsymbol{u} \cdot \nabla)\sigma + N^2 w &= 0, \\ \nabla \cdot \boldsymbol{u} &= 0, \end{aligned}$$

where $\sigma = g(\delta\rho/\rho)$ is the buoyancy acceleration and N^2 is the buoyancy frequency.

(a) Derive the internal wave equation for the pressure p, governing *slow* oscillations in a fluid when N^2 is constant.

(b) Derive a geometric relation between the group and phase velocities of the waves.

(c) Consider a vertically semi-infinite stratified fluid with constant N^2 above a sinusoidal boundary

$$z = \epsilon \sin\{k(x - Ut)\}$$

with wavenumber k, height $\epsilon \ll 1$, and travelling in the positive x-direction at speed U. Discuss the form of the motion for $N < kU$ and $N > kU$, obtaining the slope of the phase lines and the direction of energy propagation of any waves excited.