UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. *M.Sci.*

Mathematics C344: Geophysical Fluid Dynamics

COURSE CODE	:	**MATHC344**
UNIT VALUE	:	**0.50**
DATE	:	**01-MAY-02**
TIME	:	**14.30**
TIME ALLOWED	:	**2 hours**

02-C0915-3-40

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

The fluid is incompressible and has constant density ρ. Gravitational acceleration is denoted by g throughout. The Coriolis parameter is denoted by f throughout.

1. The shallow water equations can be written

$$\begin{aligned} u_t + uu_x + vu_y - fv &= -g\eta_x, \\ v_t + uv_x + vv_y + fu &= -g\eta_y, \\ H_t + (uH)_x + (vH)_y &= 0, \end{aligned}$$

where $H(x, y, t)$ is the total depth and $\eta(x, y, t)$ the free surface displacement.

(a) Show that the linearised shallow water momentum equations can be written

$$(\partial_{tt} + f^2)u = -g(\eta_{xt} + f\eta_y),$$

$$(\partial_{tt} + f^2)v = -g(\eta_{yt} - f\eta_x).$$

(b) Consider the free modes in a channel of constant width L, bounded by rigid walls at $y = 0$ and $y = L$, when the channel floor slopes linearly in the y-direction so that the undisturbed depth is given by

$$H_0(x, y) = D(1 + sy/L),$$

for constants D and s, with $s \ll 1$.
Discuss the departures of these modes from their flat-bottomed ($s = 0$) form and plot the leading order (in s) dispersion relation for the modes.
You may assume without proof that the governing equation for η of the form

$$\eta(x, y, t) = \mathcal{R}\{\bar{\eta}(y)\exp[\mathrm{i}(kx - \sigma t)]\},$$

with k and σ constants, is

$$\frac{\mathrm{d}^2\bar{\eta}}{\mathrm{d}y^2} + [(\sigma^2 - f^2)/c^2 - k^2 - fsk/(L\sigma)]\bar{\eta} = 0,$$

where $c^2 = gD$.
You may ignore any waves of frequency σ equal to $\pm f$ and the Kelvin waves of frequency σ equal to $\pm ck$.

2. The quasigeostrophic potential vorticity equation can be written

$$(\partial_t + \psi_x \partial_y - \psi_y \partial_x)(\nabla^2 \psi - F\psi + \eta_B) = 0,$$

where F is a number measuring surface deformation, $\eta_B(x, y)$ gives the shape of the lower boundary, and ψ is a streamfunction for the motion.

(a) Show that when the flow field is unbounded and the bottom slopes uniformly so that $\eta_B = \beta y$ this equation admits *finite amplitude* wave motion of the form

$$\psi = A\cos(kx + ly - \sigma t),$$

where A, k, l and σ are constants. Derive the dispersion relation for these waves.

(b) Derive and sketch a geometric relationship in wavenumber space between the group and phase velocities of waves of fixed frequency σ (and infinitesimal amplitude).

(c) Use this relationship to describe the reflection from a solid barrier at $y = 2x$ of a Rossby wave incident from $y < 2x$.

3. By considering the Navier-Stokes equations for a layer of almost-inviscid fluid (of kinematic viscosity ν) rotating rapidly (with angular speed 2Ω) about a vertical axis Oz, *derive* the Ekman compatibility condition,

$$w = \frac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}(v_x - u_y),$$

imposed on an interior flow by the Ekman layer on an impermeable, co-rotating horizontal lower boundary. Here (x, y, z) are Cartesian co-ordinates and the corresponding components of the interior velocity are (u, v, w).

4. The linearised barotropic vorticity equation for a homogeneous ocean of depth D and density ρ, forced by a surface wind stress $\boldsymbol{\tau}$ can be written

$$\nabla\psi_t + \beta\psi_x = -r\nabla^2\psi + (1/\rho D)\hat{\boldsymbol{k}}\cdot(\nabla\wedge\boldsymbol{\tau}),$$

where ψ is a streamfunction for the depth-averaged motion and $\hat{\boldsymbol{k}}$ is a vertical unit vector.

(a) Briefly identify each term in the equation.

(b) Consider the *steady* flow in the rectangular basin $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$ forced by the wind stress

$$\boldsymbol{\tau} = -\tau_0 \cos(\pi y/b)\hat{\boldsymbol{\imath}}.$$

(i) By non-dimensionalising the equations and considering the limit

$$\epsilon_S = r/\beta a \ll 1,$$

show that a Sverdrup balance holds in the bulk of the ocean. Show that, at best, a Sverdrup solution satisfies the impermeability condition at either $x = 0$ or $x = a$ but not at both simultaneously.

(ii) By considering the dynamics of a layer of thickness ϵ_S, obtain a composite form for the streamfunction in the basin and sketch the resulting flow pattern.

5. In a region sufficiently close to the equator that f can be approximated by βy, the constant-depth linearised shallow-water equations become (suitably non-dimensionalised)

$$u_t - \frac{1}{2} y v = -\eta_x,$$

$$v_t + \frac{1}{2} y u = -\eta_y,$$

$$\eta_t + u_x + v_y = 0.$$

Eliminating u and η gives the governing equation

$$v_{ttt} + \frac{1}{4} y^2 v_t - (v_{xx} + v_{yy})_t - \frac{1}{2} v_x = 0.$$

(a) Using the equation for v alone, find the dispersion relation for waves in an unbounded domain.

(b) By considering the original system, show that $v = 0$ gives a non-trivial solution.

(c) Sketch the dispersion relation, showing intersections with axes, asymptotic behaviour and turning points.

[You may use without proof the result that the solutions of the eigenvalue problem

$$D_{yy} + (\lambda - \frac{1}{4} y^2) D = 0, \quad \text{with} \quad D \to 0 \quad \text{as} \quad |y| \to \infty,$$

are the parabolic cylinder functions $D_n(y)$, with $\lambda = n + \frac{1}{2}$ for $n = 0, 1, 2, ...$]