UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C365: Geometry Of Numbers

COURSE CODE	:	**MATHC365**
UNIT VALUE	:	**0.50**
DATE	:	**09-MAY-05**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

05-C0935-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

If used, worksheets for questions 3, 4 should be tied into the answer book.

1. Define

   (a) a *lattice*, Λ, in $\mathbb{E}^n$,

   (b) a *basis* of Λ,

   (c) a *unit* lattice.

   If Λ is a unit lattice in $\mathbb{E}^2$, show that there are two points of Λ whose distance apart is not greater than $\sqrt{2/\sqrt{3}}$.

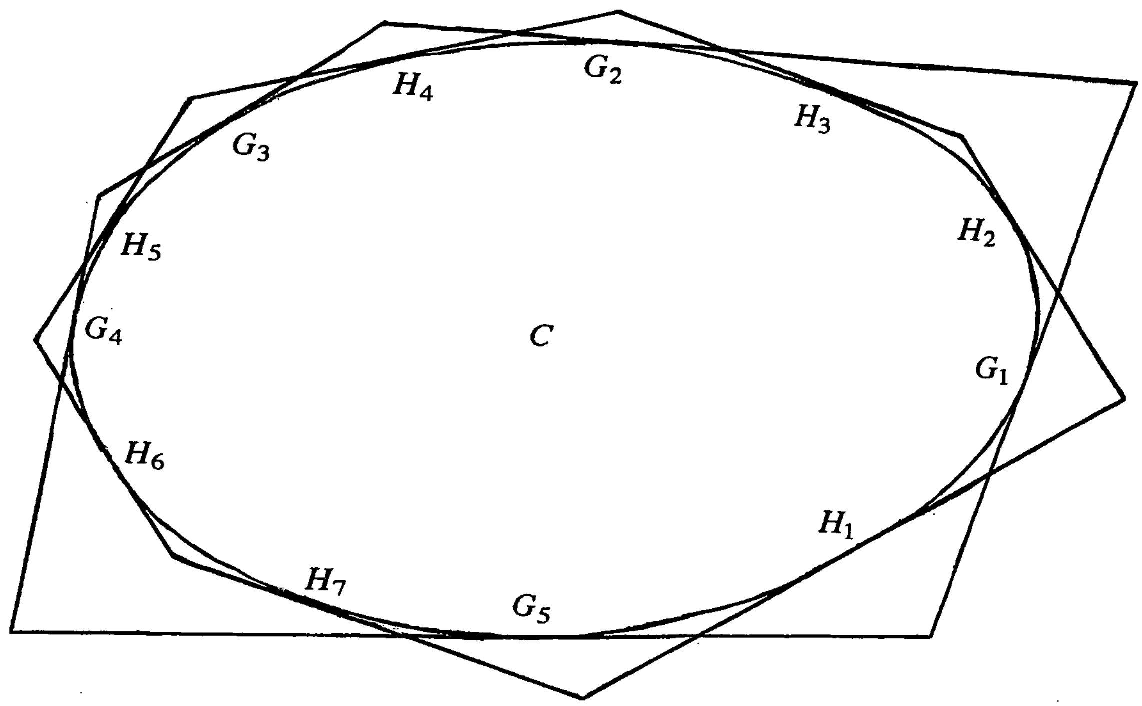
   Use this result to show that the density of a lattice packing of congruent disks in $\mathbb{E}^2$ cannot be more than $\pi/\sqrt{12}$ and show this density can be attained.

2. State and prove Minkowski's theorem.

   Let α be an irrational number. By considering the lattice $\Lambda = \{n, \alpha n - m; m, n \text{ integers}\}$ and the rectangle bounded by $x = k$, $x = -k$, $y = \frac{1}{k}$, $y = -\frac{1}{k}$, where $k > 1$, show that there are integers m, n such that $\left|\alpha - \frac{m}{n}\right| < \frac{1}{n^2}$.

3. (a) State Dowker's theorem.

(b)



The figure shows a 5-gon $G_1 \ldots G_5$ and a 7-gon $H_1 \ldots H_7$ circumscribing the convex body C. Show how the 12 edges can be rearranged to form two 6-gons about C whose total area is less than the sum of the areas of $G_1 \ldots G_5$ and $H_1 \ldots H_7$, justifying your answers.

If $n \geqslant 4$, what is the corresponding result for any $n-1$-gon G and any $n+1$-gon H circumscribing C? Use this result to deduce Dowker's theorem for C.

(c) Let K be a convex body in $\mathbb{E}^2$ with a tangent at every boundary point and no straight line segment in its boundary.

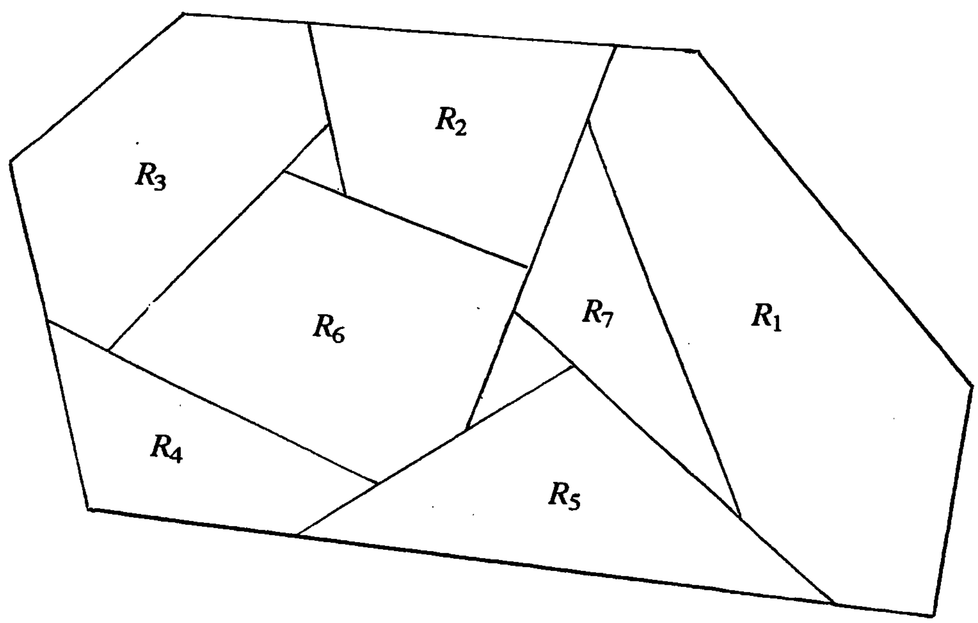
If $A_1, \ldots, A_k$ are the consecutive vertices of a k-gon of maximal area inscribed in K, show that, for $i = 1, \ldots, k$, the tangent to K at A_i is parallel to the chord $A_{i-1}A_{i+1}$, where A_{1-1} is A_k and A_{k+1} is A_1. Hence, or otherwise, show that a k-gon of maximal area inscribed in a circle is regular.

4. What is meant by describing a graph as (a) connected, (b) plane, (c) simple?

State Euler's formula for a connected, plane graph with v vertices, e edges and f faces. If this graph is also simple, show that

$$e \leqslant 3v - 6. \quad (1)$$

In $\mathbb{E}^2$, consider n convex bodies $C_1, \ldots, C_n$ with disjoint interiors contained in a hexagon H. Within H there are polygons $R_1, \ldots, R_n$ with disjoint interiors and $C_i \subseteq R_i$, $i = 1, \ldots, n$. For $i = 1, \ldots, n$, let the number of sides of R_i be s_i. Assume no R_i can be enlarged and that every point on the boundary of H belongs to either one or two R_is with exactly q points belonging to two R_is, none of these points being vertices of H.



The figure illustrates such a construction.

On this figure draw the graph with e edges constructed in Fejes Tóth's proof of his theorem and verify that, for this figure, $e > \frac{1}{2}\{(\sum_{i=1}^{n} s_i) + q - 6\}$.

Explain briefly why

$$e \geqslant \frac{1}{2}\left\{\left(\sum_{i=1}^{n} s_i\right) + q - 6\right\} \quad (2)$$

is true for every example of the described construction.

Deduce from (1) and (2) that $\sum_{i=1}^{n} s_i \leqslant 6n$.

5. What is an *affinely regular hexagon*?

If C is a convex body, what is meant by $\delta_L(C)$ and $\theta_L(C)$?

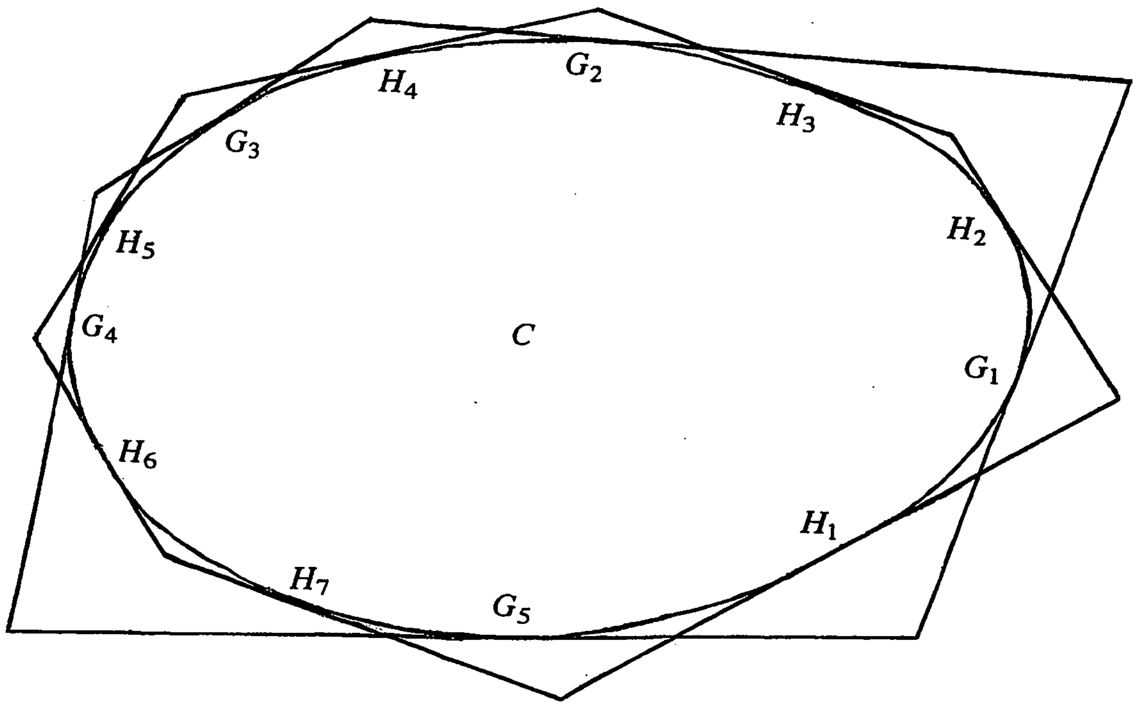
Show that a hexagon $p_1 \ldots p_6$ in $\mathbb{E}^2$ is affinely regular if and only if it is centrally symmetric and $\overrightarrow{p_2p_1} + \overrightarrow{p_2p_3} = \overrightarrow{p_3p_4}$.

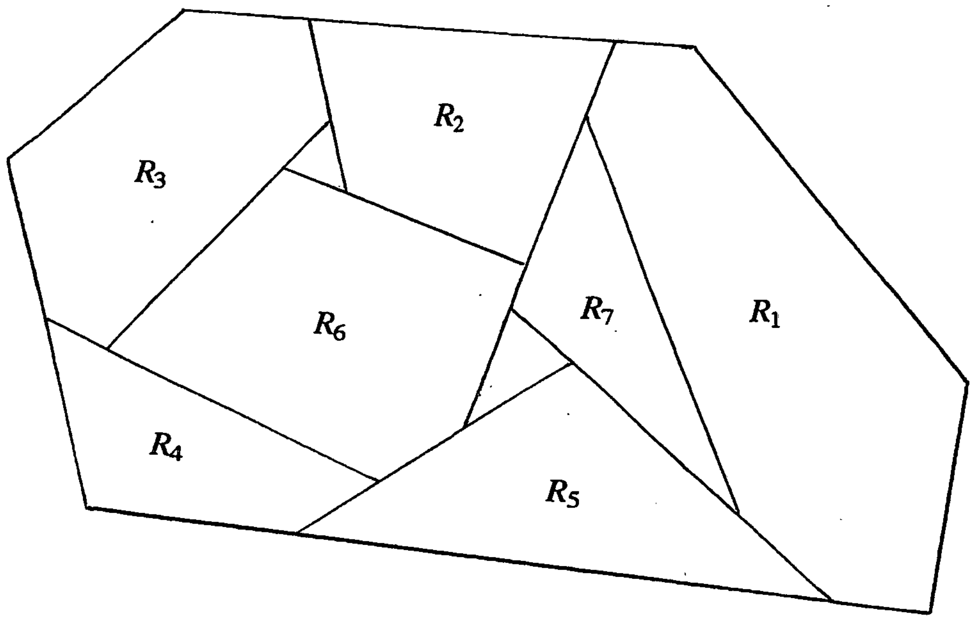
Either (a) show that any convex body C in $\mathbb{E}^2$ has an inscribed affinely regular hexagon,

or (b) assume (a) and show that, for any convex body C in $\mathbb{E}^2$,

$$\theta_L(C) \leqslant \frac{3}{2}.$$

Worksheet for Question 3. Candidate's number ....




R_3
R_2
R_6
R_7
R_1
R_4
R_5