# UNIVERSITY COLLEGE LONDON

University of London

## EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

*B.Sc.* *M.Sci.*

**Mathematics C365: Geometry Of Numbers**

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| COURSE CODE | : | **MATHC365** |
| UNIT VALUE | : | **0.50** |
| DATE | : | **19-MAY-04** |
| TIME | : | **14.30** |
| TIME ALLOWED | : | **2 Hours** |

04-C0930-3-30



**TURN OVER**

*All questions may be attempted but only marks obtained on the best* **four** *solutions will count.*

*The use of an electronic calculator is* **not** *permitted in this examination.*

*If used, worksheets for questions 3, 4, 5 should be tied into the answer book.*

1. Explain what is meant by

   (a) a *lattice*, $\Lambda$, in $\mathbb{E}^n$,

   (b) a *basis* of $\Lambda$,

   (c) a *fundamental cell* of $\Lambda$.

   For $n \geqslant 2$, let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ be a basis of $\Lambda$. Show that $\mathbf{a}_1', \ldots, \mathbf{a}_n'$ is also a basis of $\Lambda$ if, and only if,

$$[\mathbf{a}_1' \ldots \mathbf{a}_n'] = [\mathbf{a}_1 \ldots \mathbf{a}_n]\, V$$

   where the elements of the $n \times n$ matrix $V$ are integers and $\det V = \pm 1$.

2. What is meant by describing a set in $\mathbb{E}^n$ as

   (i) convex,

   (ii) a convex body,

   (iii) centrally symmetric?

   Let $C$ be a convex body in $\mathbb{E}^n$. What is meant by saying a lattice $\Lambda$

   (iv) *provides a lattice packing* of $C$,

   (v) is *admissible* for $C$?

   What is

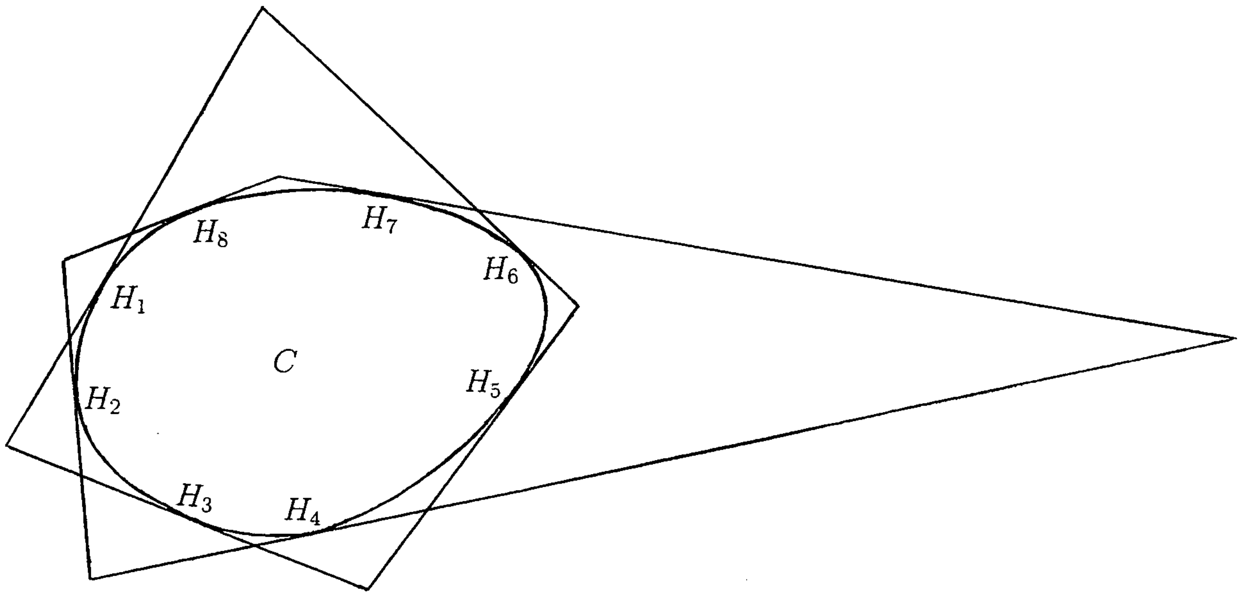
   (vi) thc *critical determinant*, $\Delta(C)$, of $C$?

   If $C$ is a convex body in $\mathbb{E}^n$, centrally symmetric about $\mathbf{0}$, prove that the lattice $\Lambda$ provides a packing of $C$ if, and only if, it is admissible for $2C$.

   Use this result to prove Minkowski's theorem in the form.

$$\Delta(C) \geqslant \frac{\text{vol} C}{2^n}.$$

3. (a) The diagram shows a convex body $C$ circumscribed by two quadrilaterals, $Q_1 : H_1H_3H_5H_6$ and $Q_2 : H_2H_4H_7H_8$, each edge being identified by its point of tangential contact with $C$. Rearrange the edges to produce two quadrilaterals of smaller total area than $Q_1$ and $Q_2$ and prove that this is so.

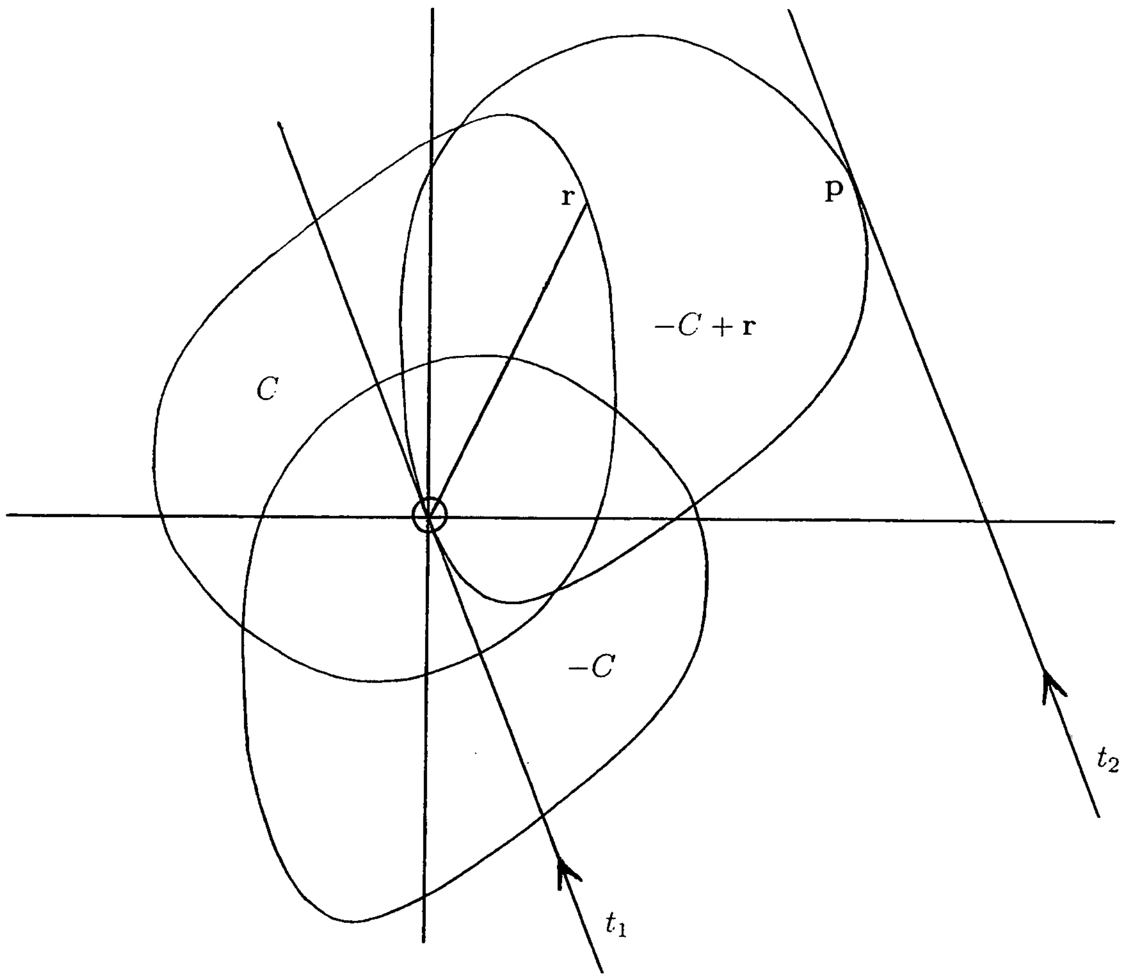



(b) Using an appropriate form of Jensen's inequality, which should be stated, show that an $n$-gon of smallest area circumscribing a disk is regular. Deduce that the smallest possible area for an $n$-gon containing the elliptical region $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leqslant 1$ is $nab \tan \frac{\pi}{n}$.

4. What is an *affinely regular* hexagon?

Show that a hexagon $p_1p_2p_3p_4p_5p_6$ is affinely regular if, and only if, it is centrally symmetric and $\overrightarrow{p_2p_1} + \overrightarrow{p_2p_3} = \overrightarrow{p_3p_4}$.

Define the *difference body* $D(C)$ of a convex body $C$ in $\mathbb{E}^2$ and show that $D(C)$ is convex and centrally symmetric in the origin.




In the drawing, $C$ is a closed convex body with a tangent, *i.e.* just one tac-line, at each boundary point. The bodies $-C$ and $-C+\mathbf{r}$, are also shown where $\mathbf{r}$ is a point on the boundary of $C$. The line $t_1$ is the tangent to $-C+\mathbf{r}$ at the origin $\mathbf{O}$ and $t_2$ is the parallel tangent on the other side of $-C+\mathbf{r}$, which touches $-C+\mathbf{r}$ at $\mathbf{p}$. Show that $\mathbf{p}$ is on the boundary of $D(C)$ and that $t_2$ is the tangent to $D(C)$ at $\mathbf{p}$.

5. (a) Suppose that $\mathbf{c}_1, \mathbf{c}_2, \ldots$ are points in $\mathbb{E}^d$ and that there is an $R > 0$ such that every ball in $\mathbb{E}^d$ of radius $R$ contains at least one $\mathbf{c}_i$. Describe how $\mathbb{E}^d$ can be divided into *Dirichlet cells*, each cell containing just one $\mathbf{c}_i$.

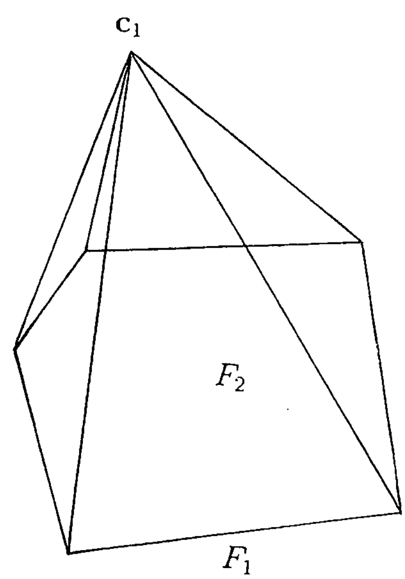
(b) What is meant by a *flat* induced by a face of a Dirichlet cell?

(c) Let the $\mathbf{c}_i$ in (a) above be the centres of unit balls which form a packing of $\mathbb{E}^d$. Let $\mathbf{x}$ be a point on a flat $F$ induced by a $(d-k)$-dimensional face of the Dirichlet cell containing $\mathbf{c}_1, 1 \leqslant k \leqslant d$. By considering the number of $\mathbf{c}_j s$, including $\mathbf{c}_1$, that are equidistant from $\mathbf{x}$ and using Blichfeldt's inequality;

for any points $\mathbf{x}, \mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$ in $\mathbb{E}^d$,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} |\mathbf{c}_i - \mathbf{c}_j|^2 \leqslant 2n \sum_{i=1}^{n} |\mathbf{x} - \mathbf{c}_i|^2 ;$$

show that the distance between $\mathbf{c}_1$ and $F$ is at least $\sqrt{\frac{2k}{k+1}}$.




(d) In the drawing, $\mathbf{c}_1$ is as in (c) ***with* d = 3** and $F_1$ is an edge of the face $F_2$ of the Dirichlet cell containing $\mathbf{c}_1$. Explain how Rogers dissected the pyramid conv $\{c_1, F_2\}$, and then the whole Dirichlet cell, into tetrahedra, each with a vertex at $\mathbf{c}_1$.

(e) For one of these tetrahedra, $\mathbf{v}_0\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3$, where $\mathbf{v}_0 = \mathbf{c}_1$, show that

$$(\mathbf{v}_k - \mathbf{v}_0).(\mathbf{v}_j - \mathbf{v}_0) \geqslant \frac{2k}{k+1}$$

for every $1 \leqslant k \leqslant j \leqslant d = 3$.



END OF PAPER