UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualification:-

B.Sc.

Mathematics C365: Geometry Of Numbers

COURSE CODE	:	**MATHC365**
UNIT VALUE	:	**0.50**
DATE	:	**23-MAY-03**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

03-C0924-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. If $f(m,n) = am^2 + 2bmn + cn^2$ is a positive definite quadratic form with $a > 0$, $ac - b^2 = 1$, show that there are integers m', n', not both 0, such that $f(m', n') \leqslant \frac{2}{\sqrt{3}}$. You may assume that any unit lattice in $\mathbb{E}^2$ has two points whose distance apart is at most $\sqrt{\frac{2}{\sqrt{3}}}$.

   Show, by giving an example, that this result is best possible.

   Given $\varepsilon > 0$, find an example of $f(m,n)$, as above, for which there are integers m, n, not both 0, such that $f(m,n) \leqslant \varepsilon$.

2. State and prove Minkowski's Theorem.

   Let p be a prime of the form $4m+1$ where m is an integer. For such a prime, there is an integer k, $0 < k < p$, such that $k^2 \equiv -1 \pmod{p}$. Using this result, without proving it, and considering the lattice generated by $(1,k)$, $(1, k+p)$, or otherwise, show that p can be expressed as the sum of the squares of two non-negative integers.

3. Let C be a convex body in the Euclidean plane and, for $n \geqslant 3$, let p_n be an n-gon of maximum area inscribed in C. Show that

$$\text{Area}(p_n) \geqslant \text{Area}(C) \cdot \frac{n}{2\pi} \sin \frac{2\pi}{n}.$$

   If the function f is convex on $[a,b]$, state Jensen's inequality for f.

   Show that no n-gon circumscribing the disk $x^2 + y^2 = r^2$ has a smaller area than the regular n-gon circumscribing this disk.

4. What is

   (i) an *affine transformation* from $\mathbb{E}^2$ to $\mathbb{E}^2$,

   (ii) an *affinely regular hexagon*?

   If $\theta_L(C)$ is the density of the thinnest lattice covering of $\mathbb{E}^2$ by the convex body C, show that $\theta_L(C) \leqslant \frac{3}{2}$.

   Show further that, if $\theta_L(C) = \frac{3}{2}$, then C is a triangle.

   You may assume without proof that C has an inscribed affinely regular hexagon.

5. Let $D(r) \geqslant 0$ be a density function which is continuous on $[0, r_0]$, $r_0 \geqslant 1$ and zero for $r > r_0$ and suppose that, for any packing $\{B^d + \mathbf{c}_i, i = 1, 2, \ldots\}$ of unit balls, where B^d is the ball of radius 1 centred at the origin, and for any $\mathbf{x} \in \mathbb{E}^d$,

$$\sum_{i=1}^{\infty} D(|\mathbf{x} - \mathbf{c}_i|) \leqslant 1.$$

   Show that the density, $\delta(B^d)$, of the densest packing of unit balls in $\mathbb{E}^d$ satisfies

$$\delta(B^d) \leqslant \frac{1}{d \int_0^{r_0} r^{d-1} D(r)\, dr}.$$

   For any points $\mathbf{x}, \mathbf{c}_1, \ldots, \mathbf{c}_n$ in $\mathbb{E}^d$, Blichfeldt's inequality is

$$\sum_{i=1}^{n} \sum_{j=1}^{n} |\mathbf{c}_i - \mathbf{c}_j|^2 \leqslant 2n \sum_{i=1}^{n} |\mathbf{x} - \mathbf{c}_i|^2.$$

   Assuming this, show that

$$\delta(B^d) \leqslant \frac{d+2}{2} 2^{-\frac{d}{2}}.$$