UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. *M.Sci.*

Mathematics C365: Geometry Of Numbers

COURSE CODE	:	**MATHC365**
UNIT VALUE	:	**0.50**
DATE	:	**08-MAY-02**
TIME	:	**14.30**
TIME ALLOWED	:	**2 hours**

02-C0920-3-50

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) State and prove Minkowski's Lattice Point Theorem.

   (b) For a real number α, consider the lattice in $\mathbb{R}^2$:

$$L = \left\{ \begin{pmatrix} n \\ \alpha n - m \end{pmatrix} : n, m \in \mathbb{Z} \right\}.$$

   Write down a basis of L and hence show that L is a unit lattice.

   For $N > 1$ consider the following rectangle:

$$R_N = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : |x| \leq N \text{ and } |y| \leq \frac{1}{N} \right\}.$$

   By calculating the area of R_N, show that R_N contains a non-zero point of L.

   Hence or otherwise show that there is a rational number $\frac{m}{n}$ with denominator $n \leq N$, such that

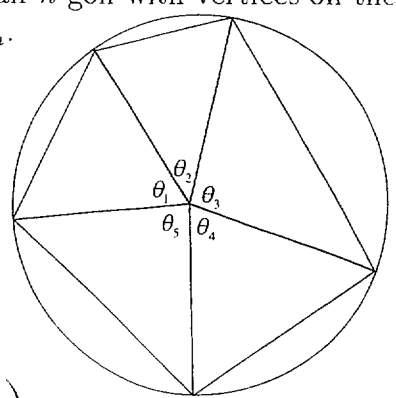
$$\left| \alpha - \frac{m}{n} \right| \leq \frac{1}{nN}.$$

2. (a) Explain what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be convex.

   Let f be a convex function. Show that

$$\frac{1}{n} \sum_{i=1}^{n} f(a_i) \geq f\left(\frac{1}{n} \sum_{i=1}^{n} a_i \right).$$

   (b) Let B denote the unit ball in $\mathbb{R}^2$. Let P be an n-gon with vertices on the boundary of B and internal angles $\theta_1, \ldots, \theta_n$.



   (i) Show that area$(P) = \frac{1}{2} \sum_{i=1}^{n} \sin(\theta)$.

   (ii) Hence deduce that area$(P) \leq \frac{n}{2} \sin\left(\frac{2\pi}{n}\right)$, with equality if P is a regular n-gon.

3. (a) Let C be a non-emptv. closed, bounded, convex subset of $\mathbb{R}^2$ such that $C = \overline{C^o}$. Sh[illegible] that [illegible]re is an affinely regular hexagon H whose vertices are on the boun[illegible]y of [illegible]

(b) Hence or otl[illegible]wise show that there is a lattice covering of $\mathbb{R}^2$ by copies of $\overline{C}$ with thickness $\leq \frac{3}{2}$.

4. (a) Define the Möbius function μ.

Prove that

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$

(b) Let $C \subset \mathbb{R}^d$ be a symmetric star body and let g denote the characteristic function of C. Define

$$f(v) = \sum_{n \in \mathbb{N}} \mu(n) g(nv), \quad v \in \mathbb{R}^d \setminus \{0\}.$$

(i) Prove that for any lattice L in $\mathbb{R}^d$, the sum

$$\sum_{l \in L \setminus \{0\}} f(l)$$

is equal to the number of primitive points of L in C.

(ii) Show that

$$\zeta(d) \int_{\mathbb{R}^d} f(v) dv = \text{vol}(C),$$

where ζ denotes the Riemann zeta function.

(iii) Hence deduce that

$$\Delta(C) \leq \frac{\text{vol}(C)}{2\zeta(d)},$$

where Δ denotes the lattice constant of C. You may assume that for any $\epsilon > 0$, there is a unit lattice L in $\mathbb{R}^d$, such that

$$\sum_{l \in L \setminus \{0\}} f(l) \leq \int_{\mathbb{R}^d} f(v) dv + \epsilon.$$

5. (a) Let $\{c_i + B^d : i \in \mathbb{N}\}$ be a packing of unit balls in $\mathbb{R}^d$. Let D be the Dirichlet cell of the ball $c_1 + B^d$ and suppose $c_1 = 0$. Consider a flag $\mathcal{F}$ of faces of D:

$$\mathcal{F} \,:\, F_0 \subset F_1 \subset \ldots \subset F_{d-1} \subset D,$$

where F_i is an i-dimensional face. Let w_i be the nearest point of F_i to 0.

Either answer part (i) or answer part (ii) but not both.

(i) Show using Blichfeldt's inequality that:

$$\langle w_{d-i}, w_{d-i} \rangle \geq \frac{2i}{i+1}.$$

(ii) By assuming the inequality of (i), prove that for $i < j$,

$$\langle w_{d-i}, w_{d-j} \rangle \geq \frac{2i}{i+1}.$$

(b) Let $D_{\mathcal{F}}$ denote the simplex with vertices $0, w_0, \ldots, w_{d-1}$.

Let S_0 be a d-dimensional simplex with vertices $0, v_1, v_2, \ldots, v_d$, such that that for $i \leq j$,

$$\langle v_i, v_j \rangle = \frac{2i}{i+1}.$$

Consider the linear map $T : D_{\mathcal{F}} \to S_0$ defined by

$$T\left(\sum_{i=1}^{d} x_i w_{d-i}\right) = \sum_{i=1}^{d} x_i v_i.$$

Prove that for $v \in B^d \cap D_{\mathcal{F}}$, we have $||T(v)|| \leq 1$.

Briefly explain why

$$\delta \leq \frac{\text{vol}(S_0 \cap B^d)}{\text{vol}(S_0)},$$

where δ denotes the density of the packing.