UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C340: Geometry And Topology

COURSE CODE	:	**MATHC340**
UNIT VALUE	:	**0.50**
DATE	:	**04-MAY-05**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

05-C0928-3-40

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Define the following notions :

   (i) chain complex ;

   (ii) the homology groups of a chain complex ;

   (iii) an exact sequence of vector spaces and linear maps.

   Explain, giving necessary definitions, but without proof, how a short exact sequence of chain complexes gives rise to a long exact sequence of homology groups.

   State the Five Lemma.

   Let

$$\begin{array}{ccccccccc} 0 \to & A_* & \xrightarrow{i} & B_* & \xrightarrow{p} & C_* & \to & 0 \\ & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 0 \to & A_* & \xrightarrow{i} & B_* & \xrightarrow{p} & C_* & \to & 0 \end{array}$$

   be a commutative diagram of chain complexes and chain maps in which both rows are exact, and β and γ induce isomorphisms in homology. Show that the induced map $\alpha_* : H_*(A) \to H_*(A)$ is also an isomorphism.

2. Explain what is meant by a simplicial complex . If K is a simplicial complex, explain further what is meant by

   i) the cone CK ;

   ii) the homology groups $H_*(K; \mathbb{F})$ where $\mathbb{F}$ is a field.

   For any finite simplicial complex K, show that the cone CK is connected, and compute the homology groups $H_r(CK; \mathbb{F})$ for $r \geqslant 0$.

   Let X_1, X_2 be subcomplexes of a finite simplicial complex X such that $X = X_1 \cup X_2$. Suppose that over a field $\mathbb{F}$ the homology of $X_1 \cap X_2$ is determined thus

$$H_r(X_1 \cap X_2) \cong \begin{cases} \mathbb{F} & r = 0 \\ \mathbb{F}^3 & r = 2 \\ 0 & \text{otherwise} \end{cases}$$

   and that both X_1 and X_2 are combinatorially equivalent to cones. Compute the homology groups $H_r(X)$ for all $r \geqslant 0$.

   [You may use without proof the Mayer Vietoris Theorem relating the homology groups $H_*(X)$, $H_*(X_1)$, $H_*(X_2)$ and $H_*(X_1 \cap X_2)$.]

3. Define the following notions :

(i) the link $\mathrm{Lk}(v, K)$ of a vertex v in a simplicial complex K ;

(ii) a simplicial surface ;

(iii) the connected sum $\Sigma_1 \# \Sigma_2$ of simplicial surfaces Σ_1, Σ_2.

If Σ_1, Σ_2 are finite connected simplicial surfaces, state and prove a relationship which holds between the Euler characteristics $\chi(\Sigma_1)$, $\chi(\Sigma_2)$ and $\chi(\Sigma_1 \# \Sigma_2)$.

Let Σ be a finite connected simplicial surface which *contains no Möbius band.* If $H_1(\Sigma; \mathbf{F}_2) \neq 0$, where $\mathbf{F}_2$ is the field with two elements, indicate the main steps involved in showing that Σ is a connected sum

$$\Sigma \sim \Sigma_1 \# T^2,$$

and deduce that $\dim_{\mathbf{F}_2} H_1(\Sigma_1; \mathbf{F}_2) = \dim_{\mathbf{F}_2} H_1(\Sigma; \mathbf{F}_2) - 2$.

4. By identifying the sides of a rectangle appropriately, explain how to obtain :

i) the torus T^2 ; ii) the Klein bottle K^2 ; iii) the Möbius band $\mathcal{M}$öb.

By means of a suitable triangulation, describe the real projective plane $\mathbb{RP}^2$, and state without proof the relation which holds between $\mathbb{RP}^2$ and $\mathcal{M}$öb.

Hence show that if Σ is a finite connected surface which contains a Möbius band then

$$\Sigma \sim \mathbb{RP}^2 \# \Sigma_1$$

for some surface Σ_1.

Give a diagrammatic proof that $\mathcal{M}\text{öb} \# T^2 \sim \mathcal{M}\text{öb} \# K^2$.

Hence deduce that $\mathbb{RP}^2 \# T^2 \sim \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$.

5. Define the *trace*, $\mathrm{Tr}(A)$, of an $n \times n$ matrix A. Explain briefly how to define the trace of a linear map $S : V \to V$, where V is a finite dimensional vector space.

Let K be a finite simplicial complex, and let $f : K \to K$ be a simplicial map. Explain what is meant by the *homological Lefschetz number* $\lambda_{\mathrm{hom}}(f)$, and state an alternative way in which this number may be calculated.

State and prove the Lefschetz Fixed Simplex Theorem.

Suppose that K is a finite simplicial complex whose rational homology is given by

$$H_r(K, \mathbb{Q}) = \begin{cases} \mathbb{Q} & r = 0, 10 \\ \mathbb{Q}^3 & r = 5 \\ 0 & \text{otherwise} \end{cases}$$

and that $T : K \to K$ is a simplicial map satisfying $T^2 = \mathrm{Id}$. Suppose also that the induced map $H_5(T) : H_5(K; \mathbb{Q}) \to H_5(K; \mathbb{Q})$ may be described by the matrix

$$H_5(T) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

List the possible values of $\lambda_{\mathrm{hom}}(T)$. Hence deduce that there exists a simplex σ of K such that $T(\sigma) = \sigma$.