UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C340: Geometry And Topology

COURSE CODE	:	**MATHC340**
UNIT VALUE	:	**0.50**
DATE	:	**09-MAY-03**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

03-C0917-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Define the following notions :

   (i) chain complex ; (ii) homology of a chain complex.

   If X is a finite simplicial complex, explain the construction of the homology groups $H_*(X; \mathbf{F})$.

   Outline the key steps involved in the computation of

   (iii) $H_*(\Delta; \mathbf{F})$ when Δ is the cone on some finite simplicial complex.

   State the Mayer-Vietoris Theorem in its geometric form, and, in this connection, show how the computation of (iii) above allows, when $n \geqslant 1$, for the computation of $H_*(S^n; \mathbf{F})$ where S^n is the standard simplicial model of the n-sphere.

2. State the Five Lemma.

   Explain, giving necessary definitions, but without proof, how a short exact sequence of chain complexes gives rise to a long exact sequence of homology groups.

   Let

$$\begin{array}{ccccccc} 0 \to & A_* \xrightarrow{i} & B_* \xrightarrow{p} & C_* \to & 0 \\ & \downarrow \alpha & \downarrow \beta & \downarrow \gamma & \\ 0 \to & A_* \xrightarrow{i} & B_* \xrightarrow{p} & C_* \to & 0 \end{array}$$

   be a commutative diagram of chain complexes and chain maps in which both rows are exact, and α and γ induce isomorphisms in homology. Show that the induced map $\beta_* : H_*(B) \to H_*(B)$ is also an isomorphism.

   Let X_1, X_2 be subcomplexes of a finite simplicial complex X such that $X = X_1 \cup X_2$, and let $\phi : X \to X$ be a simplicial map such that $\phi(X_i) \subset X_i$ for $i = 1, 2$. Suppose that

   (i) $\phi : X_1 \to X_1$ is a simplicial isomorphism, and that

   (ii) X_2 is isomorphic to a cone.

   Prove that $\phi_* : H_*(X; \mathbf{F}) \to H_*(X; \mathbf{F})$ is an isomorphism.

3. By identifying the sides of a rectangle appropriately, explain how to obtain :

   i) the torus T^2 ; ii) the Klein bottle K^2 ; iii) the Möbius band.

   Explain what is meant by the *connected sum* $M \# N$ of two surfaces M and N. Let Σ be a surface containing a Mobius band ; give a diagrammatic proof that

$$\Sigma \# T^2 \sim \Sigma \# K^2.$$

   Let X be a finite connected surface *containing no Möbius band.* If $H_1(\Sigma; \mathbf{F}_2) \neq 0$, where $\mathbf{F}_2$ is the field with two elements, indicate the main steps involved in showing that X is a connected sum

$$X \sim X_1 \# T^2.$$

   Deduce that $\dim_{\mathbf{F}_2} H_1(X_1; \mathbf{F}_2) = \dim_{\mathbf{F}_2} H_1(X; \mathbf{F}_2) - 2.$

4. Let X, Y be finite cell complexes. State the Künneth Theorem for $H_*(X \times Y; \mathbf{F})$. Hence, or otherwise, calculate

   (i) $H_*(S^2 \times \mathbf{RP}^2; \mathbf{F}_2)$ where $\mathbf{F}_2$ is the field with two elements ;

   (ii) $H_*(S^2 \times \mathbf{RP}^2; \mathbf{Q})$;

   (iii) $H_*(S^2 \times S^4 \times S^4; \mathbf{F})$.

   If K is a finite cell complex, find a relation between

   (iv) $H_n(K; \mathbf{F})$, $H_{n-2}(K; \mathbf{F})$ and $H_n(K \times S^2; \mathbf{F})$,

   and also a relation between

   (v) $\chi(K \times S^2)$ and $\chi(K)$.

5. Define the *trace*, $\mathrm{Tr}(A)$, of an $n \times n$ matrix A. Explain briefly how to define the trace of a linear map $S : V \to V$, where V is a finite dimensional vector space.

   Let K be a finite simplicial complex, and let $f : K \to K$ be a simplicial map. Explain what is meant by the *homological Lefschetz number* $\lambda_{\mathrm{hom}}(f)$, and prove that if $\lambda_{\mathrm{hom}}(f) \neq 0$ then f fixes a simplex of K.

   When K is combinatorially equivalent to $S^3 \times S^3$, show that f fixes a simplex provided also that

   i) the induced map $f_* : H_6(K; \mathbf{Q}) \to H_6(K; \mathbf{Q})$ is the identity ; and

   ii) the induced map $f_* : H_3(K; \mathbf{Q}) \to H_3(K; \mathbf{Q})$ is represented by the matrix

$$\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}.$$