UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:–

B.Sc. *M.Sci.*

Mathematics C343: Gas Dynamics

COURSE CODE	:	**MATHC343**
UNIT VALUE	:	**0.50**
DATE	:	**04–MAY–05**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

05–C0929–3–50

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

For an ideal gas the equation of state is $p = \rho RT$, and the square of the speed of sound is $a^2 = \gamma RT$, where $R = c_p - c_v$ and $\gamma = c_p/c_v$. The specific heats for an ideal gas can be regarded as constant.

For isentropic flow of an ideal gas, $p = k\rho^\gamma$ for some constant k.

1. Write down the relation between de, ds and dv required by the laws of thermodynamics, where e is specific internal energy, s is specific entropy, and v is specific volume.

   (a) Enthalpy is defined as $h = e + pv$, and the Gibbs function is defined as $g = h + Ts$. By considering $g(p, T)$, or otherwise, prove that

   $$\left(\frac{\partial s}{\partial p}\right)_T = -\left(\frac{\partial v}{\partial T}\right)_p .$$

   Given that the definition of specific heat at constant pressure is

   $$c_p = T\left(\frac{\partial s}{\partial T}\right)_p ,$$

   prove that

   $$c_p = \left(\frac{\partial h}{\partial T}\right)_p .$$

   (b) For an ideal gas, prove that h is a function of T alone by considering $h(p, T)$. Thus, given $h = 0$ when $T = 0$, prove that enthalpy and the speed of sound are related by $a = (\gamma - 1)^{1/2} h^{1/2}$.

2. The equations for one-dimensional motion of a gas are $\rho_t + (\rho u)_x = 0$, and $\rho(u_t + uu_x) = -p_x$. For isentropic flow of an ideal gas, show that the momentum equation can be written in the form

$$u_t \; + \; uu_x \; = \; - \, \alpha a a_x$$

where $\alpha = 2/(\gamma - 1)$.

The continuity equation can similarly be expressed as $\alpha a_t + a u_x + \alpha u a_x = 0$, which leads to the two characteristic equations

$$(u \pm \alpha a)_t \; + \; (u \pm a)(u \pm \alpha a)_x \; = \; 0 \; .$$

A long cylinder contains air at rest in the region $x > 0$, and has a piston at $x = 0$. From time $t = 0$ the piston is withdrawn abruptly from the cylinder with constant velocity $U = -a_0/6$, where a_0 is the speed of sound in the gas at rest. (You may assume that air is an ideal gas, and that $\gamma = 1.4$.)

(a) Prove that the c^+ characteristics that start from the position of the piston at $t > 0$ are parallel straight lines, and that the expansion fan region is bounded by the lines $x = a_o t$ and $x = (4a_o/5)t$.

(b) Sketch the path of the piston and typical c^- and c^+ characteristics in the x-t plane for $t \geq 0$.

(c) Given that the c^+ characteristics are straight lines in the expansion fan region, prove that at location $x = L$ the velocity is

$$u = -(5a_0/6)(1 - L/a_0 t)$$

for $L/a_0 \leq t \leq 5L/4a_0$.

3. An ideal gas flows steadily from a reservoir out through a nozzle with varying cross-sectional area $A(x)$. In the reservoir $\rho = \rho_0$, $p = p_0$, and effectively $u = 0$.

(a) Using $(c_pT + u^2/2)_x = 0$, prove that

$$a^2/{a_0}^2 = 2/[2+(\gamma-1)M^2]$$

where $M = u/a$ is the Mach number.

Hence show that

(i) $a^{*2}/{a_0}^2 = 2/(\gamma+1)$, and
(ii) $u^2/a^{*2} = (\gamma+1)M^2/[2+(\gamma-1)M^2]$,

where * denotes sonic conditions.

(b) For isentropic flow show that

$$\rho/\rho^* = [(\gamma+1)/(2+(\gamma-1)M^2)]^{1/(\gamma-1)} .$$

Hence obtain an expression for $(\rho u)/(\rho^* a^*)$ as a function of M. Show that this function has a maximum value of 1 when M=1. (You may assume u is positive.)

(c) Use this result to deduce that the mass flux $Q = \rho u A$ in the nozzle is at most $\rho^* a^* A_{min}$, where A_{min} is the minimum cross-sectional area in the nozzle.

4. In a steady two-dimensional flow of an ideal gas, supersonic flow with speed w_1 crosses a shock. In standard notation, the wave angle is β, and the angle of deflection is θ. After crossing the shock, the speed is w_2. The flow components normal and tangential to the shock are denoted u and v respectively; u_1 is supersonic and u_2 is subsonic.

(a) Sketch the geometry of the flow crossing the shock, indicating the angles β and θ, and the flow components u_1, v_1, w_1, u_2, v_2 and w_2.
If the upstream Mach number is $(w_1/a_1) = M_1 = 2$, what is the minimum value of β?

(b) From the momentum equations the jump conditions are $\rho_1 {u_1}^2 + p_1 = \rho_2 {u_2}^2 + p_2$ and $\rho_1 u_1 v_1 = \rho_2 u_2 v_2$. State the jump condition that can be derived from the equation of continuity, and hence deduce that $v_1 = v_2$ and that

$$u_1 + RT_1/u_1 = u_2 + RT_2/u_2 \ .$$

(c) The energy equation gives the jump condition $h_1 + {w_1}^2/2 = h_2 + {w_2}^2/2$. Use this condition to show that

$$2{a_1}^2 + (\gamma - 1){w_1}^2 = 2{a_2}^2 + (\gamma - 1){w_2}^2 = (\gamma + 1)a^{*2} \ ,$$

where a^* is the speed of sonic flow.

(d) Using the above conditions and results, derive Prandtl's relation for an oblique shock:

$$u_1 u_2 = a^{*2} - {v_1}^2(\gamma - 1)/(\gamma + 1) \ .$$

5. For steady irrotational isentropic flow of an ideal gas the governing equations can be written $\nabla.(\rho \underline{u}) = 0$ and $\nabla(\frac{1}{2}\underline{u}^2 + \frac{\gamma}{\gamma-1}(p/\rho)) = 0$.

(a) A two-dimensional flow has $p = p_0$, $\rho = \rho_0$ and $\underline{u} = (u_0, 0)$ when undisturbed, where u_0 is a positive constant. For small perturbations it can be shown that $\tilde{p}/p_0 = \gamma\tilde{\rho}/\rho_0$, where $\tilde{p}$ etc. denote perturbations to a flow that is undisturbed far upstream. Derive the linearised relations

(i) $\rho_0(\tilde{u}_x + \tilde{v}_y) + u_0\tilde{\rho}_x = 0$, and

(ii) $u_0\tilde{u} + {a_0}^2\tilde{\rho}/\rho_0 = 0$.

(b) The velocity perturbation can be expressed as $\underline{\tilde{u}} = \nabla\phi$. For supersonic flow, show that

$$\lambda^2\phi_{xx} - \phi_{yy} = 0 ,$$

where $\lambda^2 = ({u_0}^2/{a_0}^2) - 1$.

(c) A thin aerofoil with top surface $y = T(x)$ and bottom surface $y = B(x)$, for $0 \le x \le L$, is placed in this uniform supersonic flow. (Here $T(0) = B(0) = 0$.) In which region of the (x, y) plane is the pressure unperturbed? Provide a sketch to illustrate your answer.

The linearised boundary conditions for $\tilde{v}$ are $\tilde{v}{=}u_0\, dT/dx$ on $y{=}0^+$, and $\tilde{v}{=}u_0\, dB/dx$ on $y{=}0^-$. Show that

$$\lambda\tilde{p}/(\rho_0{u_0}^2) = dT/dx$$

on the top surface, and find an equivalent expression for $\tilde{p}$ on the lower surface. Deduce that the lift on the aerofoil is zero if $T(L) = B(L) = 0$.