UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C343: Gas Dynamics

COURSE CODE	:	**MATHC343**
UNIT VALUE	:	**0.50**
DATE	:	**16-MAY-03**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

03-C0918-3-40

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (i) State the first and second laws of thermodynamics. Distinguish between reversible and irreversible processes, and define carefully the quantities you introduce.

   (ii) Stefan's law relates the internal energy E of a system to its volume V and temperature T by

$$E = \sigma V T^4$$

where σ is a constant. Show that, if the zero of the entropy S and that of the absolute temperature T are taken to coincide, then

$$S = 4E/(3T)$$

and the pressure p is given by

$$p = E/(3V)$$

2. Gas is moving in the x-direction across a normal shock and the pressure, density and speed of the gas change from the constant values p_1, ρ_1 and u_1 when x is large and negative, to the constant values p_2, ρ_2 and u_2 when x is large and positive. The one-dimensional equations of continuity, momentum and energy are

$$\frac{d}{dx}(\rho u) = 0,$$

$$\rho u \frac{du}{dx} = \frac{d}{dx}\left(-p + (\lambda + 2\mu)\frac{du}{dx}\right),$$

$$\rho u \frac{d}{dx}\left(\frac{\mathcal{R}T}{\gamma - 1} + \frac{p}{\rho} + \frac{1}{2}u^2\right) = \frac{d}{dx}\left((\lambda + 2\mu)u\frac{du}{dx} + k\frac{dT}{dx}\right),$$

where k, λ, μ are coefficients of conduction and viscosity and T is the temperature, determined by the perfect gas law $T = p/\mathcal{R}\rho$.

Write down first integrals of these equations and deduce (but do not solve) equations which relate p_2, ρ_2 and u_2 to p_1, ρ_1 and u_1.

If $k = \gamma\mathcal{R}(\lambda + 2\mu)/(\gamma - 1)$, show that

$$\frac{\gamma p}{(\gamma - 1)\rho} + \frac{1}{2}u^2 = D,$$

and that

$$\frac{\lambda + 2\mu}{\rho_1 u_1}\frac{du}{dx} = \frac{\gamma + 1}{2\gamma}u - C + \frac{(\gamma - 1)D}{\gamma u}$$

where the constants C and D may be written in terms of p_1, ρ_1, u_1.

3. A tube of variable cross-section $A(x)$ has $A(x) \to \infty$ as $x \to -\infty$ where a steady adiabatic isentropic flow of a perfect gas satisfies reservoir conditions

$$u(x) \to 0, \quad \rho(x) \to \rho_0, \quad p(x) \to p_0, \quad a(x) \to a_0.$$

Starting with Bernoulli's equation in the form

$$\frac{1}{2}u^2 + \frac{a^2}{\gamma - 1} = \text{constant},$$

show that

$$u(x) = \overline{a}(1 - y^n)^{\frac{1}{2}}$$

where

$$\overline{a} = \left(\frac{2}{\gamma - 1}\right)^{\frac{1}{2}} a_0, \quad n = \frac{\gamma - 1}{\gamma} \quad \text{and} \quad y = p(x)/p_0.$$

Write down the equation of continuity appropriate to this flow and show that the mass flux Q, defined by $Q = \rho u A$, is a constant. Calculate

$$f(y) = \frac{Q}{\rho_0 \overline{a} A(x)} \quad \text{and} \quad M^2 = \frac{u^2}{a^2}$$

as functions of y. Show that $f(y)$ has a maximum value of

$$\left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} \left(\frac{\gamma-1}{\gamma+1}\right)^{\frac{1}{2}} \quad \text{at} \quad y = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{n}}.$$

Illustrate these results by a sketch, showing that $M^2 = 1$ at this maximum.

A wind tunnel with a supersonic working section has a single throat of area A_t. The flow is smooth and shock free. Find Q in terms of A_t. If $\gamma = 5/3$ and the Mach number M has the value 2 at the exit where $x = x_e$, show that

$$\frac{A(x_e)}{A_t} = \frac{49}{32}.$$

4. Express the equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \qquad \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0,$$

$$p = k\rho^{\gamma}, \qquad a^2 = \gamma p/\rho,$$

governing the one-dimensional motion of a perfect gas moving isentropically in a uniform tube in characteristic form

$$\left(\frac{\partial}{\partial t} + (u \pm a)\frac{\partial}{\partial x}\right)\left(u \pm \frac{2a}{\gamma - 1}\right) = 0.$$

Here $u(x,t)$ is the velocity of the gas, $a(x,t)$ is the speed of sound, x measures distance along the tube, t is the time and γ is the ratio of the specific heats.

Gas is at rest in the region $x > 0$, so that $u(x,0) = 0$ and $a(x,0) = a_0$. At $t = 0$, a piston, initially at $x = 0$, starts to withdraw with increasing speed in the x decreasing direction. Show that

(i) the density at the piston will fall to zero when its withdrawal speed is $2a_0/(\gamma - 1)$;

(ii) if $x_1 (< 0)$ is the position of the piston at the time t_1 at which its withdrawal speed is $2a_0/(\gamma + 1)$, then $a(x_1, t)$ is constant for $t > t_1$.

5. Assuming Bernoulli's equation and the equation of continuity for an isentropic flow of a perfect gas in the form

$$\frac{1}{2}(u^2+v^2)+\frac{a^2}{\gamma-1}=\frac{a_\infty^2}{\gamma-1}+\frac{1}{2}U_\infty^2,$$

$$\frac{\partial}{\partial x}\left(a^{\frac{2}{\gamma-1}}u\right)+\frac{\partial}{\partial y}\left(a^{\frac{2}{\gamma-1}}v\right)=0,$$

derive the equation

$$(a^2-u^2)\frac{\partial^2\phi}{\partial x^2}-2uv\frac{\partial^2\phi}{\partial x\partial y}+(a^2-v^2)\frac{\partial^2\phi}{\partial y^2}=0$$

satisfied by the potential ϕ for a steady irrotational motion in two dimensions.

If the flow is approximately uniform, with a speed U_∞ in the direction of x increasing, show that the potential $\phi_1=\phi-U_\infty x$ satisfies the linear equation

$$(M_\infty^2-1)\frac{\partial^2\phi_1}{\partial x^2}=\frac{\partial^2\phi_1}{\partial y^2}$$

where M_∞ is the Mach number of the uniform stream. Derive the linearised form of the boundary condition on a surface $y=\varepsilon f(x)$ where $|\varepsilon|\ll 1$.

Air flows in the xy-plane in the positive x-direction over an infinite corrugated surface $y=\varepsilon\sin kx$. Determine the perturbation potential $\phi_1(x,y)$ in both the subsonic and supersonic situations. Comment on the flow at large distances from the surface in each case.