UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C329: Functions Of A Complex Variable I

COURSE CODE	:	**MATHC329**
UNIT VALUE	:	**0.50**
DATE	:	**03-MAY-05**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

05-C0923-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Define the maximal function $M(r)$.
   (b) State Schwarz's lemma.
   (c) Use the maximum principle to prove the following generalisation of Schwarz's lemma: if f is a holomorphic function from the unit disc $B(0,1)$ into itself, and
   $$f(0) = f'(0) = \ldots = f^{(n-1)}(0) = 0,$$
   then $M(r) \leq r^n$.

2. Let $\mathcal{F}$ be a family of functions defined on a domain D. Let A be an arbitrary subset of D.
   (a) Define what it means that
      (i) $\mathcal{F}$ is uniformly bounded on A
      (ii) $\mathcal{F}$ is equicontinuous on A
      (iii) $\mathcal{F}$ is normal.
   (b) State and prove Weierstrass' theorem.
   (c) Show that there is a largest domain D on which
   $$\mathcal{F} = \{ne^{-nz} : n = 1, 2, \ldots\}$$
   is normal, and find D.

3. (a) Define what the order of growth of an entire function means.
   (b) Define the canonical factors $E_k(z)$, and state Hadamard's product theorem.
   (c) Show that the Hadamard product of $\cos z$ is
   $$\prod_{n=0}^{\infty} \left(1 - \frac{4z^2}{\pi^2(2n+1)^2}\right).$$
   You may use without proof that the order of growth of $\cos z$ is 1.

4. Consider the family of holomorphic functions $f(z)$ on the unit disc $B(0,1)$ of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \qquad (*)$$

   (a) Show that there exists a constant $c > 0$ such that the image of any holomorphic function f given by $(*)$ on the unit disc $B(0,1)$ contains a disc of radius c. (You may use the theorem that there exists a constant $c' > 0$ such that for any M and for any such function $f(z)$, if $|f(z)| \leq M$ then the image f contains a disc around 0 of radius c'/M.)

   (b) Show that for any $0 \neq w \in \mathbb{C}$ there is a holomorphic function $f(z)$ on $B(0,1)$ of the form $(*)$ whose image does not contain w.

   (c) Deduce that one cannot always choose the centre of the disc in the statement (a) to be 0.

5. (a) Define Euler's constant γ and the function $\Gamma(z)$.

   (b) State Gauss' formula and use it to prove that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \quad z \notin \mathbb{Z}.$$

   You may use the formula $\sin \pi z = \pi z \prod_{k=1}^{\infty}(1 - \frac{z^2}{k^2})$ without proof.

   (c) Calculate $|\Gamma(\frac{1}{2} + it)|$ for $t \in \mathbb{R}$.