UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C329: Functions Of A Complex Variable I

COURSE CODE	:	**MATHC329**
UNIT VALUE	:	**0.50**
DATE	:	**11-MAY-04**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

04-C0920-3-30

TURN OVER

All *questions may be attempted but only marks obtained on the best* **four** *solutions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

1. State what is meant by a bilinear transformation. Show that the bilinear transformation $w = \frac{1}{z}$ maps a circle not passing through the origin onto a circle. Find the general form of the bilinear transformation mapping $|z| < 1$, conformally onto $|w - 2| < 1$.

2. (a) Define what is meant by the holomorphic function $f(z)$ having a <u>natural barrier</u>.

   By differentiation, show that the function $f(z) = \sum_{n=0}^{\infty} (2^n + 1)^{-1} z^{2^n+1}$ has $|z| = 1$ as a natural barrier.

   (b) The function $f(z) = \sum_{n=0}^{\infty} (-1)^n z^n$ is defined for $|z| < 1$ with analytic continuation $F(z) = \frac{1}{1+z}$ to $\mathbb{C} \setminus \{-1\}$. By expanding $f(z)$ about a non-real point a with $|a| < 1$, obtain a Taylor series expansion $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ for $f(z)$. For which values of z does this converge? Show that in fact it converges to the function $F(z)$.

3. Suppose that the function $f(z)$ is holomorphic for $|\arg z| \leqslant \dfrac{\pi}{2\alpha}$, $\alpha > \frac{1}{2}$ with $|f(z)| \leqslant M$ for $|\arg z| = \dfrac{\pi}{2\alpha}$. By considering $F(z) = \exp(-\varepsilon\, z^{\alpha}) f(z)$ for $\varepsilon > 0$ show that either

(a) $|f(z)| \leqslant M$ for $|\arg z| \leqslant \dfrac{\pi}{2\alpha}$, or

(b) $\displaystyle\limsup_{r\to\infty} \frac{\log M(r)}{r^{\alpha}} > 0$,

where $M(r) = \max\left\{|f(z)| : |z| \leqslant r,\ |\arg z| \leqslant \dfrac{\pi}{2\alpha}\right\}$.

Explain why (b) can be strengthened to assert that

$$\liminf_{r\to\infty} \frac{\log M(r)}{r^{\alpha}} > 0.$$

Any ancillary result required need not be proved but should be clearly stated.

4. (a) Define the class S of normalized univalent functions and show that if $f(z) \in S$ then also $\phi(z) = [f(z^2)]^{1/2} \in S$. Show that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$ then $|a_n| < en$ for all $n \geqslant 2$. You may assume that, if $f \in S$ then

$$\int_0^{2\pi} \left|f(re^{i\theta})\right| d\theta < \frac{2\pi r}{1-r} \quad \text{for } 0 < r < 1.$$

(b) Show that the polynomial $p(z) = z + a_2 z^2 \in S$ if and only if $|a_2| \leqslant \frac{1}{2}$.

5. (a) Show that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is univalent and $f(z) \neq \gamma$ for $|z| < 1$ then $|\gamma| \geqslant \frac{1}{4}$. When is equality attained? You may assume that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is univalent then $|a_2| \leqslant 2$.

(b) If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is holomorphic for $|z| < 1$ (but not necessarily univalent) and $|f'(z)| \leqslant 1$ for $|z| < 1$ show that again, if $f(z) \neq \gamma$ for $|z| < 1$, then $|\gamma| \geqslant \frac{1}{4}$.

MATHC329 END OF PAPER