UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc.   *M.Sci.*

Mathematics C329: Functions Of A Complex Variable I

COURSE CODE	:	**MATHC329**
UNIT VALUE	:	**0.50**
DATE	:	**30-APR-03**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

03-C0912-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. The function $f(z)$ is holomorphic inside and on a closed contour C except for one point P on C. Moreover, $|f(z)| \leqslant M$ for $z \in C$ except for $z = P$. Suppose that we can find a function $w(z)$, holomorphic and non-zero inside C such that, for every $\epsilon > 0$ there is a sequence $\{C_n\}$ of crosscuts of C with dist $(P, C_n) \longrightarrow 0$ as $n \longrightarrow \infty$ such that $|w(z)^{\varepsilon} f(z)| \leqslant M$ on C_n. Show then that $|f(z)| \leqslant M$ for z inside C.

   Suppose now that C has length l and $|f(z)| \leqslant M$ on C. Use Cauchy's Integral Formula for $[f(z)]^n$ to show that, for z inside C,

$$|f(z)|^n \leqslant KM^n$$

   for a suitable constant K independent of n. Deduce that $|f(z)| \leqslant M$ inside C.

2. Define the order ρ and lower order λ of an entire function $f(z)$. Show that $z^{-\frac{1}{2}} \sin\left(z^{-\frac{1}{2}}\right)$ is an entire function and find its order and lower order.

   Show that an entire function of order ρ can have at most 2ρ distinct radial asymptotic values. Any ancillary result used should be clearly stated but need not be proved.

3. Define the class S of univalent functions in the unit disk $\Delta = \{z : |z| < 1\}$.
   If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$, use the fact that $|a_2| \leqslant 2$ to prove the following assertions

   (a) If $f(z) \neq \gamma$ for $z \in \Delta$ then $|\gamma| \geqslant \frac{1}{4}$.

   (b) If $f(z) \neq \alpha$ and $\neq \beta$ and $\beta \neq \alpha$ than $\left|\frac{1}{\alpha} - \frac{1}{\beta}\right| \leqslant 4$.

   Show that the function $f(z) = \frac{z}{1-z^2}$ is in S and determine the image of Δ under f. Use this to show that in (b) the bound '4' cannot be replaced by any smaller number.

4. State what is meant by a mapping $w = f(z)$ being *conformal* at a point $z = z_0$.

Show that every bilinear transformation mapping the half-plane $y \geqslant 0$ onto $|w| \leqslant 1$ is of the form

$$w = e^{i\lambda}\left(\frac{z-\alpha}{z-\overline{\alpha}}\right) \qquad \lambda \in \mathbf{R}, \quad \operatorname{Im}\alpha > 0,$$

where $z = x + iy$.

Find the general form of the conformal mapping of the first quadrant

$$\{z = x + iy \,:\, x^2 + y^2 \leqslant 1,\ x \geqslant 0,\ y \geqslant 0\}$$

onto the closed disk $|w| \leqslant 1$.

5. (a) What is meant by a function having a closed curve C as a *natural barrier*. Give an example, with proof, of a function $f(z)$ holomorphic in $\Delta = \{z : |z| < 1\}$ which has $|z| = 1$ as a natural barrier.

(b) State Parsevals' Identity. The function $f(z)$ is holomorphic in Δ with $f(0) = 0$ and $|f'(0)| \geqslant 1$. If $|f'(z)| \leqslant M$ in Δ and $f(z) \neq \gamma$ in Δ, show that

$$|\gamma| \geqslant \frac{1}{4M}.$$