UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C336: Functional Analysis

COURSE CODE	:	**MATHC336**
UNIT VALUE	:	**0.50**
DATE	:	**03-MAY-06**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

06-C1013-3-40

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) State the Hahn-Banach theorem for real normed spaces. Prove it for finite-dimensional spaces.

   (b) Show that, given any element x in a normed space V, the following identity holds:
$$\|x\| = \sup_{f \in V^* \setminus \{0\}} \frac{|f(x)|}{\|f\|}.$$

2. (a) State the Baire category theorem.

   (b) State and prove the principle of uniform boundedness.

   (c) Show that the principle of uniform boundedness is not valid in incomplete normed spaces.

   (d) Let U and V be Banach spaces. Let $A_n, A \in B(U, V)$ and $x_n, x \in V^*$. Suppose that $s\text{-}\lim_{n\to\infty} A_n = A$ and $\lim_{n\to\infty} x_n = x$. Show that $A_n x_n$ converges strongly to Ax.

3. (a) State the Banach open mapping theorem.

   (b) State and prove the closed graph theorem. You may use the open mapping theorem without proof.

   (c) Let V_1 and V_2 be closed linear subspaces of a Banach space V and let $A : V \to V$ be a bounded linear operator. Suppose that $A(V_1) = V_2$. Show that there exists a constant C such that for each $y \in V_2$ there exists $x \in V_1$ such that $\|x\| \leq C\|y\|$ and $Ax = y$.

4. (a) Let V be a Banach space and $A : V \to V$ be a bounded operator. Define the notions *resolvent set, spectrum, point spectrum, discrete spectrum* of A.

   (b) Prove that the spectrum of A is a closed subset of the complex plane.

   (c) Let $A : l^2 \to l^2$ be an operator defined by $A\mathbf{e}^1 = 0$, $A\mathbf{e}^j = \mathbf{e}^{j-1}$ $(j = 2, 3, 4, \dots)$, where $\mathbf{e}^j$ are vectors of the standard basis of l^2. Find the eigenvalues and the spectrum of A.

5. (a) State the Banach contraction mapping theorem.

   (b) Let A be a map from the complete non-empty metric space into itself such that A^k is a contraction for some $k \in \mathbb{N}$. Show that there exists a unique fixed point for A.

   (c) Use this result to prove that the equation

$$f(t) + \int_0^t \left(\cos(s^2 - st + t^2) + \sin(ts)\right) f(s) ds = 5t^4 \qquad (0 \le s, t \le 1)$$

has a unique solution $f \in C[0, 1]$.

END OF PAPER