UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C336: Functional Analysis

COURSE CODE	:	**MATHC336**
UNIT VALUE	:	**0.50**
DATE	:	**18-MAY-05**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

05-C0927-3-40

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) State the Hahn-Banach theorem for normed spaces.

   (b) Show that, given any non-zero element x in a normed space V, there exists an element $f \in V^*$ such that $f(x) = ||x||$ and $||f|| = 1$.

   (c) (i) Prove that if $x_n \in V$ is a sequence such that

   $$w\text{-}\lim_{n\to\infty} x_n = x$$

   and there exists

   $$\lim_{n\to\infty} \|x_n\|,$$

   then

   $$\lim_{n\to\infty} ||x_n|| \geq ||x||.$$

   (ii) Give an example which show that it can happen that

   $$\lim_{n\to\infty} ||x_n|| > ||x||.$$

2. (a) State the Baire category theorem.

   (b) State and prove the principle of uniform boundedness.

   (c) Let V be a Banach spaces and U be a reflexive Banach space. Let $A_n, A \in B(U, V)$ and $f_n, f \in V^*$. Suppose that $s\text{-}\lim_{n\to\infty} A_n = A$ and $w\text{-}\lim_{n\to\infty} f_n = f$. Define the functionals $f_n A_n, fA \in U^*$ by $f_n A_n(x) = f_n(A_n(x))$, $fA(x) = f(A(x))$, $x \in U$. Show that $f_n A_n$ converges weakly to fA.

3. (a) State the Banach open mapping theorem.

   (b) State and prove the closed graph theorem.

   (c) Let V_1 and V_2 be closed linear subspaces of a Banach space V and suppose that V is an algebraic direct sum of V_1 and V_2, i.e. given any $x \in V$, there exists a unique representation $x = x_1 + x_2$, where $x_i \in V_i$. Define a map $P : V \to V$ by $x = x_1$. Show that P is linear and bounded.

4. (a) Let V be a Banach space and $A : V \to V$ be a bounded operator. Define the notions *resolvent set, spectrum, discrete spectrum* and *essential spectrum* of A.

   (b) Prove that the spectrum of A is a closed subset of the complex plane.

   (c) Let $A : l^1 \to l^1$ be an operator defined by $A\mathbf{e}^j = \mathbf{e}^{j+1}$ $(j = 1, 2, \dots)$ where $\mathbf{e}^j$ are vectors of the standard basis of l^1. Find the discrete spectrum and the spectrum of A.

5. (a) State the Banach contraction mapping theorem.

   (b) Let A be a map from the complete metric space into itself such that A^k is a contraction for some $k \in \mathbb{N}$. Show that there exists a unique fixed point for A.

   (c) Use this result to prove that the equation

$$f(t) + \frac{4}{3} \int_0^t \cos(s + t) f(s) ds = e^t \qquad (0 \le s, t \le 1)$$

   has a unique solution $f \in C[0, 1]$.

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