UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C336: Functional Analysis

COURSE CODE	:	**MATHC336**
UNIT VALUE	:	**0.50**
DATE	:	**05-MAY-04**
TIME	:	**10.00**
TIME ALLOWED	:	**2 Hours**

04-C0924-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) State the Hahn-Banach theorem for normed spaces.

   (b) Show that, given any non-zero element x in a normed space V, there exists an element $f \in V^*$ such that $f(x) = ||x||$ and $||f|| = 1$.

   (c) Let U, V be normed linear spaces, and $A \in B(V, U)$ be a bounded operator. Define the operator $A^* : U^* \to V^*$ by the formula:

$$(A^* f)(x) = f(Ax) \qquad (x \in V, f \in U^*).$$

   Prove that A^* is a continuous linear operator, and $||A^*|| = ||A||$.

2. (a) State and prove the principle of uniform boundedness (you can assume the Baire category theorem).

   (b) Suppose that $\{\alpha_k\}$ is a sequence of real numbers such that the series $\sum_{k=1}^{\infty} \alpha_k x_k$ converges whenever $\{x_k\}$ is a sequence of real numbers such that $\sum_{k=1}^{\infty} x_k^2 < \infty$. Using the above result or otherwise show that $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$.

3. (a) State the Banach open mapping theorem.

   (b) Prove that if U, V are Banach spaces, and $A \in B(U, V)$ is a bijection, then $A^{-1} \in B(V, U)$.

   (c) Let X be a closed subspace of $C[0, 2]$. Suppose for every $f \in C[0, 1]$ there is an $F \in X$ whose restriction to $[0, 1]$ is f. Show that there is a constant C such that the function F can always be chosen to satisfy $||F||_{C[0,2]} \le C||f||_{C[0,1]}$.

4. (a) State and prove the closed graph theorem (you can assume the open mapping theorem).

   (b) Let V_1 and V_2 be closed linear subspaces of a Banach space V and suppose that V is an algebraic direct sum of V_1 and V_2, i.e. given any $x \in V$, there exists a unique representation $x = x_1 + x_2$, where $x_i \in V_i$. Define a map $P : V \to V$ by $Px = x_1$. Show that P is linear and bounded.

5. (a) State and prove the Baire category theorem. You must prove all auxiliary statements you use.

   (b) Prove that the set A consisting of polynomials with rational coefficients is of the first category in $C[0, 1]$.

END OF PAPER