UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C336: Functional Analysis

COURSE CODE : **MATHC336**

UNIT VALUE : **0.50**

DATE : **19-MAY-03**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

03-C0916-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. a) Give the definitions of a base and a subbase for a topology.

   b) Define the product topology on a product space and prove that the product of any family of compact topological spaces is compact. You may assume Alexander's Subbase Theorem.

   c) Let X be a normed linear space. Prove that the unit ball

$$B_1^* = \{F \in X^* : \|F\| \leq 1\}$$

   in the dual X^* of a normed linear space X is w^*-compact.

2. a) Let $F_1, \ldots, F_n$ and F be linear functionals on a vector space X over $\mathbb{R}$ and let

$$N = \{x : F_1 x = \cdots = F_n x = 0\}.$$

   Prove that the following properties are equivalent:

   (i) There are scalars $\alpha_1, \ldots, \alpha_n$ such that

$$F = \sum_{i=1}^{n} \alpha_i F_i.$$

   (ii) There is a $\lambda \in \mathbb{R}$ such that

$$|Fx| \leq \lambda \max\{|F_i x| : i = 1, \ldots, n\}$$

   for all $x \in X$.

   (iii) $Fx = 0$ for all $x \in N$.

   b) Prove that, if F is a linear functional on the dual X^* of a normed linear space which is continuous in the weak*-topology, then

$$F \in \Phi[X],$$

   where $\Phi : X \to X^{**}$ is the canonical embedding.

3. Let A and B be disjoint nonempty convex sets in a real normed linear space X.

   a) If A has an interior point, then there is an $F \in X^*$, $F \neq 0$, and $\gamma \in \mathbb{R}$ such that
   $$F(a) \leq \gamma \leq F(b)$$
   for all $a \in A$ and for all $b \in B$.

   b) If A and B are either both open or have $\operatorname{dist}(A, B) > 0$, then there is an $F \in X^*$ and $\gamma \in \mathbb{R}$ such that
   $$F(a) < \gamma < F(b)$$
   for all $a \in A$ and for all $b \in B$.

4. Let K be a weakly compact subset of a real Banach space X and let $E(K)$ denote the set of extreme points of K. Let K have the relative topology induced by the weak topology on X. Suppose $x_0 \in E(K)$ and U is an open subset of K containing x_0.

   a) Prove that there is an $F \in X^*$ and $\gamma > 0$ such that
   $$x_0 \in \{x \in K : F(x) > \gamma\} \subset \{x \in K : F(x) \geq \gamma\} \subset U.$$

   b) Prove that if U_n, $n = 1, 2, \ldots$, are relatively open dense subsets of $E(K)$, then $\bigcap_{n=1}^{\infty} U_n$ is dense in $E(K)$.

5. a) State the Hahn-Banach Theorem.

   b) If M is a subspace of a real normed linear space X and if $f \in M^*$, prove that f can be extended to an $F \in X^*$ such that $\|f\| = \|F\|$.

   c) Let X be a real normed linear space and $x \in X$, $x \neq 0$. Prove that there exists an $F \in X^*$ such that $Fx = \|x\|$ and $\|F\| = 1$.

   d) Let X be an infinite dimensional real normed linear space.
   Let $S = \{F \in X^* : \|F\| = 1\}$ be the unit sphere in the dual space X^* and let $B = \{F \in X^* : \|F\| \leq 1\}$ be the unit ball. Prove that S is dense in B in the *weak** topology.