UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. *M.Sci.*

Mathematics C336: Functional Analysis

COURSE CODE	:	**MATHC336**
UNIT VALUE	:	**0.50**
DATE	:	**16-MAY-02**
TIME	:	**14.30**
TIME ALLOWED	:	**2 hours**

02-C0914-3-40

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. a) State the Open Mapping Theorem and give a proof of the special case when the range is finite dimensional.

   b) Let X, Y be Banach spaces, Z a normed linear space, $S \in \mathcal{L}(X,Y)$, $T \in \mathcal{L}(X,Z)$. Use (a) to prove that if S is surjective and $\ker S \subseteq \ker T$, then there exists a unique $R \in \mathcal{L}(Y,Z)$ such that $T = R \circ S$.

2. a) State and prove the Uniform Boundedness Principle (i.e., Banach-Steinhaus Theorem).

   b) Prove that if (T_n) is a sequence of continuous linear maps of a Banach space X into a real normed linear space Y such that $\lim_{n\to\infty} T_n x$ exists for all $x \in X$, then the map $T : X \to Y$ defined by

$$Tx = \lim_{n\to\infty} T_n x$$

   is continuous.

3. a) Let X^* be the dual of a real normed linear space X. Prove that if a linear functional f on X^* separates a non-empty weak* open set B from another set A in X^* (i.e. there exists $\lambda \in \mathbb{R}$ such that $Fb \leq \lambda \leq Fa$ for all $b \in B$, $a \in A$), then f is weak* continuous.

   b) Prove that if A is a weak* closed convex subset of X^* and $F_0 \in X^* \backslash A$, then there is a weak* continuous $f \in X^{**}$ and $\gamma \in \mathbb{R}$ such that

$$fF_0 < \gamma < fF \quad \text{for all } F \in A.$$

   c) Prove that the unit ball in the dual X^* of a real normed space X is the weak* closed convex hull of its extreme points.

4. a) If Y is a proper closed linear subspace of a normed space X, and $\epsilon > 0$, prove that there is a $x_\epsilon \in X$, $\|x_\epsilon\| = 1$, such that

$$\inf\{\|y - x_\epsilon\| : y \in Y\} > 1 - \epsilon.$$

Hint: If x is a point not in Y and $\epsilon > 0$, choose a point in Y within $(1 + \epsilon)$ times the distance between x and Y from x.

b) Prove that if X is infinite dimensional, then the closed unit ball in X has an open cover without a finite subcover.

5. A Banach limit is any bounded linear functional L on ℓ_∞ such that for $x = (x_1, x_2, \ldots) \in \ell_\infty$,

i) $L(x) \geq 0$ if $x_n \geq 0$ for all n,

ii) $L(x) = L(\sigma x)$, where $\sigma(x) = (x_2, x_3, \ldots)$,

iii) $L(x) = 1$ if $x = (1, 1, 1, \ldots)$.

Prove that

a) if L is a Banach limit, then $\underline{\lim}\, x_n \leq L(x) \leq \overline{\lim}\, x_n$ for all $x \in \ell_\infty$ (Hint: note that it follows from ii) that it suffices to prove that $\inf x_n \leq L(x) \leq \sup x_n$);

b) Banach limits exist. (Hint: define $l : c \to \mathbb{R}$ by $l(x) = \lim_{n\to\infty} x_n$ and consider the function $p : \ell_\infty \to \mathbb{R}$ defined by $p(x) = \overline{\lim}\, \frac{x_1 + \ldots + x_n}{n}$.)

END OF PAPER