UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C356: Fractal Geometry

COURSE CODE	:	**MATHC356**
UNIT VALUE	:	**0.50**
DATE	:	**15-MAY-06**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

06-C1017-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Give the definition of a similitude $w : \mathbb{R}^2 \to \mathbb{R}^2$.

   (b) Show that if $w : \mathbb{R}^2 \to \mathbb{R}^2$ is a similitude, $w(x) = Ax + b$, where $b \in \mathbb{R}^2$ and A is a 2×2 matrix, then A can be written either as $A = rR_\theta$ or as $A = rRR_\theta$, where R is a reflection and R_θ is a rotation by an angle θ, $0 \le \theta < 2\pi$.

   (c) Verify that the following two functions are similitudes and find their scaling factors:
$$w_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} .4 & .3 \\ .3 & -.4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$w_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} .24 & .07 \\ -.07 & .24 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

   (d) Find the fractal dimension of the attractor of the hyperbolic iterated functions system $\{\mathbb{R}^2; w_1, w_2\}$.

2. Let A be a non-empty compact subset of a complete metric space X.

   (a) Give the definition of the fractal dimension $D(A)$ of A.

   (b) Prove that if $0 < r < 1$, $\epsilon_n = r^n$ and $D = \lim_{n\to\infty} \frac{\ln(\mathcal{N}(A,\epsilon_n))}{-\ln \epsilon_n}$ exists, then D is the fractal dimension of A. Here $\mathcal{N}(A, \epsilon_n)$ is the smallest number of closed balls of radius ϵ_n that cover the set A.

   (c) Give the definition of the Hausdorff-Besicovitch dimension of a subset of $\mathbb{R}^m$. Prove that the Hausdorff-Besicovitch dimension of any countable set is zero.

   (d) Compute the fractal dimension of the compact set $\{1, 0, 1/2, 1/3, \ldots, 1/n, \ldots\}$ in $\mathbb{R}$.

3. (a) Let $\{X; w_1, w_2, \ldots, w_N\}$ be a hyperbolic IFS. Give the definition of the code space associated with the IFS and the metric on the code space.

   (b) Let $\{X; w_1, \ldots, w_N\}$ be a hyperbolic IFS. For σ in the code space Σ, $n \in \mathbb{N}$, and $x \in X$ let $\phi(\sigma, n, x) = w_{\sigma_1} \circ \ldots \circ w_{\sigma_n}(x)$ and $\phi(\sigma) = \lim_{n\to\infty} \phi(\sigma, n, x)$. An address of a point a in the attractor A of the IFS is any element of the set $\phi^{-1}(a) = \{\omega \in \Sigma : \phi(\omega) = a\}$. Prove that a point $x \in A$ is a periodic point of the IFS if and only if it has a periodic address.

(c) Define what it means for the IFS to be totally disconnected, just-touching and overlapping.

(i) Prove that a hyperbolic IFS $\{X; w_1, w_2, \dots, w_N\}$ is totally disconnected if and only if $w_i(A) \cap w_j(A) =$ for all $i, j \in \{1, 2, \dots, N\}$.

(ii) Give an example in $\mathbb{R}$ of an IFS that is just touching. Justify your example.

4. Let $\mathbb{C}$ denote the complex plane.

(a) Consider a set A in $\mathbb{C}$ that is constructed by replacing the square with vertices $0, 1, i, 1+i$ by four squares of $\frac{1}{5}$ the side length positioned inside the square in the corners, then replacing these four boxes each by four boxes of $\frac{1}{5}$ their side lengths, and iterating this process. Construct a hyperbolic IFS whose attractor is A and compute the fractal dimension of A.

(b) Prove that if $f : \mathbb{C} \to \mathbb{C}$ is a continuous function and $z_0 \in \mathbb{C}$, then $\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists if and only if there exists a similitude of the form $w(z) = az + b$, where $a, b \in \mathbb{C}$, such that $\lim_{z \to z_0} \frac{|f(z)-w(z)|}{|z-z_0|} = 0$.

(c) Find a similitude that approximates the behaviour of $f(z) = (z-1)^3$ near $z_0 = 1 - i$.

5. Let X denote a compact metric space and $\mathcal{H}(X)$ the space of non-empty closed subsets of X.

(a) Define the Hausdorff metric h on $\mathcal{H}(X)$ and prove that for every $\epsilon > 0$ and every $E \in (\mathcal{H}(X), h)$ there exists a finite subset F of E such that $\mathcal{H}(F, E) < \epsilon$.

(b) Prove that a decreasing sequence (K_n) of non-empty compact sets in X is a Cauchy sequence in $(\mathcal{H}(X), h)$ with limit $\bigcap_{n=1}^{\infty} K_n$.