UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:–

B.Sc. *M.Sc.i.*

Mathematics C356: Fractal Geometry

COURSE CODE	:	**MATHC356**
UNIT VALUE	:	**0.50**
DATE	:	**11–MAY–05**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

05–C0933–3–30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Let X denote a complete metric space and $\mathcal{F}(X)$ the space of non-empty closed bounded subsets of X.

   (a) Define the Hausdorff metric h on $\mathcal{F}(X)$ and prove that $(\mathcal{F}(X), h)$ is complete.

   (b) Prove that if (E_n) is a Cauchy sequence in $(\mathcal{F}(X), h)$, then it converges to the set $E = \{x \in X : \exists$ a Cauchy sequence (x_n) with $x_n \in E_n$ such that $\lim_{n\to\infty} x_n = x\}$.

2. (a) Give the definition of a similitude $w : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

   (b) Show that the following maps are similitudes and find their scaling factors:

$$w_1\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 2 & \frac{3}{2} \\ \frac{3}{2} & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{3}{2} \end{pmatrix}$$

$$w_2\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{25}\begin{pmatrix} 6 & -\frac{7}{4} \\ \frac{7}{4} & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{3}{2} \end{pmatrix}$$

$$w_2\begin{pmatrix} x \\ y \end{pmatrix} = \frac{\sqrt{2}}{4}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

   (c) Show that the IFS $\{\mathbb{R}^2; w_1, w_2, w_3\}$ is totally disconnected.

   (d) Find the fractal dimension of the attractor of the hyperbolic iterated functions system $\{\mathbb{R}^2; w_1, w_2, w_3\}$.

3. Let A be a non-empty compact subset of a complete metric space X.

   (a) Give the definition of the fractal dimension $D(A)$ of A.

   (b) Prove that if $D = \lim_{n\to\infty} \frac{\ln(\mathcal{N}(A, 2^{-n}))}{\ln(2^n)}$ exists, then D is the fractal dimension of A. Here $\mathcal{N}(A, 2^{-n})$ is the smallest number of closed balls of radius 2^{-n} that cover the set A.

   (c) Let $A \subset \mathbb{R}^d$, cover $\mathbb{R}^d$ by boxes of side length 2^{-n} that just touch at the sides, and let $\mathcal{N}_n(A)$ denote the number of boxes that intersect A. If A is a non-empty compact subset of $\mathbb{R}^d$ with fractal dimension $D(A)$, show that $D(A) = \lim_{n\to\infty} \frac{\ln(\mathcal{N}_n(A))}{\ln(2^n)}$.

4. Consider a set A in $\mathbb{R}^2$ that is constructed by replacing the square with vertices $(0,0),(1,0),(0,1),(1,1)$ by five squares of $\frac{1}{3}$ the side length positioned inside the square with four in the corners and one in the centre, then replacing these five boxes each by five boxes of $\frac{1}{3}$ their side lengths in the same way, and iterating this process. Construct a just-touching hyperbolic IFS whose attractor is A and compute the fractal dimension of A.

5. Let $\{X; w_1, w_2, \ldots, w_N\}$ be a hyperbolic IFS on a complete metric space X.

   (a) Give the definition of the code space associated with the IFS and the metric on the code space.

   (b) Let K be a non-empty compact subset of X. Show that there exists a compact subset K' of X such that $K \subset K'$ and $w_n : K' \to K'$ for $n = 1, 2, \ldots, N$.

   (c) For σ in the code space Σ, $n \in \mathbb{N}$, and $x \in X$ let $\phi(\sigma, n, x) = w_{\sigma_1} \circ \ldots \circ w_{\sigma_n}(x)$ and $\phi(\sigma, x) = \lim_{n\to\infty} \phi(\sigma, n, x)$. Show that the convergence is uniform on compact subsets of X, and that $\phi(\sigma, x) = \phi(\sigma, y)$ for any x and y in X.