UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C356: Fractal Geometry

COURSE CODE	:	**MATHC356**
UNIT VALUE	:	**0.50**
DATE	:	**29-MAY-03**
TIME	:	**10.00**
TIME ALLOWED	:	**2 Hours**

03-C0922-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Let (X, d) denote a metric space and $\mathcal{F}(X)$ the space of non-empty closed bounded subsets of X.

   (a) Define h on $\mathcal{F}(X)$ by

   $$h(A, B) = sup\{d(x, B), d(y, A) : x \in A, y \in B\}.$$

   Prove that h is a metric on $\mathcal{F}(X)$.

   (b) If (X, d) is compact, prove that $(\mathcal{F}(X), h)$ is compact.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a contraction map.

   (a) Show that f has exactly one fixed point.

   (b) Prove that $(f^n([0, 1]))_{n=0}^{\infty}$ is a Cauchy sequence in the space of non-empty closed bounded subsets of $\mathbb{R}$ with the Hausdorff metric h, and show that it converges to the fixed point of f.

   (c) Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{1}{4}x + \frac{1}{2}$. Show that f is a contraction, find its fixed point and show directly that $(f^n([0, 1]))_{n=0}^{\infty}$ converges to this fixed point . Show that $(f^n(\mathbb{R}))_{n=0}^{\infty}$ does not converge to the fixed point of f in any reasonable sense.

3. (a) Give the definition of a similitude $w : \mathbb{R}^2 \to \mathbb{R}^2$ and the definition of a Möbius transformation on $\hat{\mathbb{C}}$.

(b) Show that a Möbius transformation that maps ∞ to ∞ is of the form $f(z) = az + b$, where $a, b \in \mathbb{C}$ and $a \neq 0$, and show that this is a similitude not involving a reflection.

(c) Verify that the following two functions are similitudes and find their scaling factors:

$$w_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.6 & 0.4 \\ 0.4 & -0.6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$w_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.4 & 0.3 \\ 0.3 & -0.4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

(d) Find the fractal dimension of the attractor of the hyperbolic iterated functions system $\{\mathbb{R}^2; w_1, w_2\}$.

4. Let A be a non-empty compact subset of a complete metric space X.

(a) Give the definition of the fractal dimension $D(A)$ of A.

(b) Describe the attractor of the IFS

$$\{\mathbb{C}; w_1 = \frac{1}{10}z, w_2 = \frac{1}{10}z + \frac{9}{10}, w_3 = \frac{1}{10}z + \frac{9}{10}i, w_4 = \frac{1}{10}z + (\frac{9}{10} + \frac{9}{10}i)\}$$

by drawing the first two iterations starting from the square with vertices $0, 1, i, 1+i$ and compute its fractal dimension.

(c) Give the definition of the Hausdorff-Besicovitch dimension of a subset of $\mathbb{R}^m$.

(d) Let $A = \{0, 1, 1/n : n \in \mathbb{N}\}$ in $\mathbb{R}$. Show that the Hausdorff-Besicovitch dimension of A is not equal to the fractal dimension of A.

5. (a) Let $\{X; w_1, w_2, \ldots, w_N\}$ be a hyperbolic IFS. Give the definition of the code space associated with the IFS and the metric on the code space.

(b) For a totally disconnected hyperbolic IFS $\{X; w_1, \ldots, w_N\}$ give the definition of the shift dynamical system $\{S; A\}$ associated with $\{X; w_1, \ldots, w_N\}$. Describe how it is equivalent to the dynamical system $\{\Sigma_N : T\}$, where $T(\sigma_1\sigma_2\ldots) = (\sigma_2\sigma_3\ldots)$.

(c) Consider the IFS $\{\mathbb{C}; w_1, w_2, w_3\}$, where $w_1(z) = \frac{3}{7}z$, $w_2(z) = \frac{3}{7}z + 1$, and $w_3(z) = \frac{3}{7}z + i$. Let $\phi : \Sigma_3 \to A$ be the associated code space map and let $a_0 = \phi(1311\overline{322})$. Find the orbit of a_0 and show that it converges to a repulsive fixed point of S.