UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc.

Mathematics C356: Fractal Geometry

COURSE CODE	:	**MATHC356**
UNIT VALUE	:	**0.50**
DATE	:	**01-MAY-02**
TIME	:	**10.00**
TIME ALLOWED	:	**2 hours**

02-C0918-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Give the definition of a Möbius transformation on $\hat{\mathbb{C}}$.

   (b) Show that the most general Möbius transformation, which maps ∞ to ∞, is of the form $f(z) = az + b$, $a, b \in \hat{\mathbb{C}}$, $a \neq 0$, and show that this is a similitude.

   (c) Show that any Möbius transformation that is not a similitude may be written

   $$h(z) = e + \frac{f}{z+g}$$

   for some e, f, $g \in \hat{\mathbb{C}}$, $f \neq 0$.

   (d) Interpret the Möbius transformation $f(z) = i + \frac{1}{z-i}$ in terms of operations on the sphere.

2. (a) Give the definition of a just-touching *IFS* $\{X; w_n : n = 1, 2, \ldots, N\}$.

   (b) Consider the *IFS* $\{[0,1]; w_n(x) = \frac{n}{10} + \frac{1}{10}x : n = 0, 1, \ldots, 9\}$ and the associated code space using the symbols $\{0, 1, 2, \ldots, 9\}$. Show that the attractor of the IFS is $[0,1]$ and that it is just-touching. Identify the points with multiple addresses.

3. (a) Define the Hausdorff distance h on the space of non-empty compact subsets of $\mathbb{R}^d$.

   (b) Prove that for every non-empty compact set $C \subset \mathbb{R}^d$ and any $\epsilon > 0$ there is a finite set $F \subset C$ such that $h(F, C) < \epsilon$.

   (c) Prove that an increasing sequence C_n of non-empty compact subsets of $\mathbb{R}^d$ converges, in the Hausdorff distance, to the closure of the set $\bigcup_{k=0}^{\infty} C_k$ provided that the latter set is bounded.

4. (a) Give the definition of Hausdorff dimension $\dim_H C$ of a subset C of $\mathbb{R}^d$.

(b) Show that for any subset C of $\mathbb{R}^d$

$$\dim_H C \leq \liminf_{n\to\infty} \left\{ \frac{\ln(\mathcal{N}_n(C))}{\ln(2^n)} \right\},$$

where $\mathcal{N}_n(C)$ denotes the number of boxes of side length $\frac{1}{2^n}$ that intersect C.

(c) Show that for any countable subset C of $\mathbb{R}^d$, $\dim_H C = 0$.

5. Let $f_0, f_1 : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_0(x) = \frac{x}{9}, \quad f_1(x) = \frac{x+2}{3}.$$

(a) Show that $\{f_0, f_1\}$ is a hyperbolic IFS and $0, 1 \in A \subset [0,1]$, where A is the attractor of $\{f_0, f_1\}$.

(b) Show that $\{f_0, f_1\}$ is a totally disconnected IFS.

(c) Let Σ be the code space and $\phi : \Sigma \to A$ the coding map of $\{f_0, f_1\}$. Let $\Sigma_1 = \{\sigma = (\sigma_i)_1^\infty \in \Sigma : \sigma_1 = 1\}$ and show that

$$\phi(\Sigma_1) \subset \left[\frac{2}{27}, \frac{1}{9}\right] \cup \left[\frac{8}{9}, 1\right].$$

(d) Give the definition of the fractal dimension $\dim_F C$ of a subset C of $\mathbb{R}^d$. Show that for the set A in part (a) that

$$\dim_F A = \frac{\ln 2}{\ln 3} - \frac{\ln(\sqrt{5}-1)}{\ln 3}.$$

END OF PAPER