UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M231: Fluid Mechanics

COURSE CODE	:	**MATHM231**
UNIT VALUE	:	**0.50**
DATE	:	**17-MAY-04**
TIME	:	**10.00**
TIME ALLOWED	:	**2 Hours**

04-C0959-3-100

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count. In all questions the fluid is incompressible and inviscid and has constant density ρ. Gravitational acceleration is denoted by g throughout.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Define the terms stream line, particle path, and streak line. A fluid moves two-dimensionally so that its velocity $\boldsymbol{u}$ at time t is given by

$$\boldsymbol{u} = e^t \boldsymbol{i} + e^{-t} \boldsymbol{j} \ .$$

Obtain equations in terms of the Cartesian co-ordinates (x, y) for the following:

(i) the stream line through $(1, 1)$ at $t = 0$;

(ii) the particle path of a particle released from $(1, 1)$ at $t = 0$;

(iii) the streak line at $t = 0$ formed from all particles released from $(1, 1)$ at times $t \leqslant 0$.

Sketch the three lines on the same diagram in the (x, y)-plane.

(b) The velocity field of a fluid is given by $\boldsymbol{u} = y^2 \boldsymbol{i} + x^3 \boldsymbol{j}$.

(i) Show that this represents a rotational flow of an incompressible fluid.

(ii) Find a velocity potential for the flow or give a reason why none can exist.

(iii) Find a stream function for the flow or give a reason why none can exist.

2. An isotropic line source of strength $2\pi m$ at the origin is in a uniform stream of speed U. Take the x-axis to be in the direction of the flow at large distance and write down both a velocity potential and a stream function for this flow .

(a) Show that there is a stagnation point upstream of the source and find its position.

(b) As the flow approaches the stagnation point from upstream it splits to pass either side of a stream line through the stagnation point. Find an equation for the dividing stream line.

(c) Find the points of intersection of the dividing stream line with the axes. Show that far downstream all stream lines are parallel to the x-axis.

(d) Calculate the far-downstream separation of the two branches of the dividing stream line. Explain the result physically.

(e) Sketch the motion. Distinguish between fluid from the source and fluid from upstream.

3. A line vortex of strength $2\pi\kappa$ is at $z = b$ and has complex potential given by $-i\kappa \log(z-b)$. The vortex lies outside the cylinder $|z| = a$ (where a, b, κ are real, $a < b$ and $\kappa > 0$).

   (a) State (without proof) the Circle Theorem. Show that the complex potential for this system can be written

$$w(z) = -i\kappa \log(z-b) + i\kappa \log(\frac{a^2}{z} - b) \ .$$

   (b) Describe the image system inside the cylinder.

   (c) Show that the image system induces a velocity at the vortex ($z = b$) in the negative-y direction of magnitude:

$$\frac{\kappa a^2}{b(b^2-a^2)} \ .$$

   (d) Suppose that the vortex is free to move and does so under the sole influence of the image system. Describe its motion and state how this would change if κ were negative.

4. A homogeneous, incompressible, inviscid fluid is flowing steadily under the action of a conservative body force $\boldsymbol{F}$. The body force has potential G per unit mass so that $\boldsymbol{F} = -\boldsymbol{\nabla} G$.

   (a) By considering without proof the Euler equations $\rho\frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla} p + \rho \boldsymbol{F}$, or otherwise, show that if $\boldsymbol{\omega} = \boldsymbol{\nabla} \wedge \boldsymbol{u}$ is the vorticity then

$$\boldsymbol{\nabla}\left(\frac{p}{\rho} + \frac{1}{2}\boldsymbol{u}^2 + G\right) = \boldsymbol{u} \wedge \boldsymbol{\omega} \ ,$$

   [You may also use without proof the identity $(\boldsymbol{u} \,.\boldsymbol{\nabla})\boldsymbol{u} + \boldsymbol{u} \wedge \boldsymbol{\omega} = \boldsymbol{\nabla}\left(\frac{1}{2}\boldsymbol{u}^2\right)$.]

   (b) Hence deduce that $\frac{p}{\rho} + \frac{1}{2}\boldsymbol{u}^2 + G$ is constant along any stream line.

   (c) Fluid of density ρ is flowing along a horizontal pipe of variable cross-section. At two points A and B on the centre-line of the pipe, pressure measurements are taken and found to be p_A and p_B respectively. Show that if $\Delta p = p_A - p_B$, and a_1 and a_2 are, respectively, the cross-sectional areas of the pipe at A and B, then the velocity at point B is equal to

$$\left[\frac{2a_1^2 \Delta p}{\rho(a_1^2 - a_2^2)}\right]^{\frac{1}{2}} .$$

5. A small-amplitude wave is propagating in the positive x-direction on the surface of water of infinite depth. The co-ordinate y is measured vertically upwards from the undisturbed free surface and the displacement of the surface is given by

$$\eta = a\cos(kx - \omega t) \ ,$$

where a, k, ω are positive real constants. One boundary condition (at $y = 0$) on the velocity potential ϕ is then

$$\frac{\partial \phi}{\partial t} = -g\eta \ .$$

(a) By considering the motion of a particle on the free surface, show that another boundary condition on ϕ at $y = 0$ is

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} \ .$$

(b) State the third boundary condition on ϕ.

(c) Find the velocity potential and show that

$$\omega^2 = gk \ .$$

(d) Suppose that a wave, given by $\eta = a\cos(kx + \omega t)$, of the same amplitude propagating in the opposite direction is also present.

(i) Write down the velocity potential for the combined motion.

(ii) Determine the total horizontal component of the fluid velocity at any point in the fluid.

(iii) Deduce the surface elevation for a wave in the region $x > 0$ when there is a rigid vertical wall at $x = 0$.

The following identities may be useful:

$$\cos A + \cos B = 2\cos\left(\frac{1}{2}(A+B)\right)\cos\left(\frac{1}{2}(A-B)\right) \ ,$$

$$\sin A - \sin B = 2\cos\left(\frac{1}{2}(A+B)\right)\sin\left(\frac{1}{2}(A-B)\right) \ .$$

6. Consider a shallow stream flowing steadily along a channel of constant width. Let the channel floor rise slowly by a small amount $k > 0$. Let the fluid depth upstream of the rise be h_1, downstream of the rise be h_2, and the rise of the surface downstream (relative to the upstream surface) be r.

   (a) Use (without proof) Bernoulli's equation to show that

   $$H(h_1) = H(h_2) + k ,$$

   where $H(h) = h + \frac{Q^2}{2gh^2}$ and where Q is a constant which is proportional to the mass flux along the channel.

   (b) Sketch the graph of $H(h)$ obtaining the position h_m of the minimum of H (for $h > 0$). Define the terms Froude number, subcritical, and supercritical. Indicate on the graph the subcritical and supercritical regions.

   (c) Show that

   $$r = \frac{Q^2}{2g}(h_1^{-2} - h_2^{-2}) .$$

   (d) Describe the difference in the behaviour of the flow depending on whether h_1 is greater or less than h_m.