UNIVERSITY COLLEGE LONDON

*University of London*

**EXAMINATION FOR INTERNAL STUDENTS**

*For The Following Qualifications:-*

*B.Sc. M.Sci.*

**Mathematics M231: Fluid Mechanics**

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| COURSE CODE | : | **MATHM231** |
| UNIT VALUE | : | **0.50** |
| DATE | : | **29-MAY-03** |
| TIME | : | **14.30** |
| TIME ALLOWED | : | **2 Hours** |

03-C0952-3-80


**TURN OVER**

*All questions may be attempted but only marks obtained on the best* **four** *solutions will count.*
*The use of an electronic calculator is* **not** *permitted in this examination.*

The fluid is incompressible and inviscid and has constant density $\rho$. Gravitational acceleration is denoted by $g$ throughout.

1. (a) Define the terms streamline, particle path and streakline. A fluid moves two-dimensionally so that its velocity $\boldsymbol{u}$ is given by

$$\boldsymbol{u} = (1+t)\boldsymbol{i} + \boldsymbol{j},$$

where $\boldsymbol{i}$ and $\boldsymbol{j}$ are the unit vectors for the Cartesian coordinates $(x, y)$. Obtain equations in terms of $x$ and $y$ alone for the following:

(i) the streamline through (1,1) at time $t = 0$,
(ii) the particle path for a particle released from (1,1) at time $t = 0$,
(iii) the streakline through (1,1) formed by particles released from (1,1) at times $t \leqslant 0$.

Sketch the three loci on the same diagram in the $(x, y)$ plane.

(b) The velocity field of fluid is given by

$$\boldsymbol{u} = (-3y^2\boldsymbol{i} + 2x\boldsymbol{j})/(x^2 + y^3).$$

(i) Show that this represents rotational flow of an incompressible fluid.
(ii) Find a velocity potential for the flow or give a reason why none exists.
(iii) Find a streamfunction for the flow or give a reason why none exists.

2. A stationary circular cylinder with boundary $x^2 + y^2 = a^2$ is in a two-dimensional irrotational flow field whose velocity has Cartesian components $(u, v)$ such that

$$u - \lambda y \to 0 \qquad \text{and} \qquad v - \lambda x \to 0 \qquad \text{as} \qquad (x^2 + y^2)/a^2 \to \infty.$$

Here $\lambda$ is a positive constant and the circulation around the cylinder is zero.

(a) Derive a streamfunction for the flow.

(b) Show that the greatest value of the fluid speed on the cylinder is $2\lambda a$.

(c) Show that the velocity vanishes at four points on the cylinder.

(d) Sketch streamlines for the flow. (You may find it convenient to first sketch streamlines for the flow in the absence of the cylinder)

You may use without proof the relation

$$\boldsymbol{u} = -\boldsymbol{i}_z \times \nabla\psi,$$

between streamfunction $\psi$ and velocity $\boldsymbol{u}$, and the form

$$\nabla\psi = \frac{\partial\psi}{\partial r}\boldsymbol{i}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\boldsymbol{i}_\theta + \frac{\partial\psi}{\partial z}\boldsymbol{i}_z,$$

where $\boldsymbol{i}_r$, $\boldsymbol{i}_\theta$, $\boldsymbol{i}_z$ are the unit vectors of a cylindrical polar co-ordinate system.

You may also find useful the form

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}.$$

3. The circle theorem states that the complex velocity potential for potential flow outside the solid cylinder $|z| = a$, of radius $a > 0$, can, in certain circumstances, be written in the form

$$w(z) = f(z) + \overline{f}(a^2/z).$$

(a) Give sufficient conditions for the validity of the theorem, defining $f$ and $\overline{f}$ and proving that there is no normal flow through the circle $|z| = a$.

(b) Use the circle theorem to write down the complex potential for a source of strength $2\pi m$ at $z = b$ outside the cylinder $|z| = a$ where $a$ and $b$ are real and $a < b$. Sketch the streamlines of the flow and describe the image system inside the cylinder in terms of elementary singularities.

4. Homogeneous incompressible inviscid fluid flows *steadily* under the action of a conservative body force $\mathbf{F}$ with potential $G$ per unit mass (so $\mathbf{F} = -\nabla G$).

   (a) Show that
   $$\nabla(p/\rho + \frac{1}{2}\boldsymbol{u}^2 + G) = \boldsymbol{u} \wedge \boldsymbol{\omega},$$
   where $\boldsymbol{\omega} = \nabla \wedge \boldsymbol{u}$ is the vorticity.

   (b) Deduce that $p/\rho + \frac{1}{2}\boldsymbol{u}^2 + G$ is constant along any streamline.

   (c) A funnel is in the form of a cone of semi-angle $\alpha$ and is placed with its vertex downwards. It is filled with water and then the water is allowed to flow out of the funnel through a *small* hole at the vertex. If the exit stream of water has a cross-section of area $A$, find the velocity of the fluid in the stream when the depth of the water in the funnel is $h$ and the rate at which the water level is decreasing is equal to $U$. Show further that (to leading order in the smallness of the exit hole)
   $$U = \frac{A}{\pi \tan^2 \alpha}(\frac{2g}{h^3})^{\frac{1}{2}}.$$
   (You may assume that the fluid motion is approximately steady).

You may use without proof the Euler equations
$$\rho\frac{D\boldsymbol{u}}{Dt} = -\nabla p + \rho\mathbf{F},$$
and the identity
$$\boldsymbol{u}.\nabla\boldsymbol{u} + \boldsymbol{u} \wedge \boldsymbol{\omega} = \nabla(\frac{1}{2}\boldsymbol{u}^2).$$

5. A small-amplitude wave is progressing in the positive $x$-direction on the surface of water of constant density $\rho$ and infinite depth, so that the equation of the surface is $y = \eta(x,t)$ where $y$ is measured vertically upwards from the undisturbed surface. The linearised dynamic boundary condition at the surface, $y = 0$, can be written

$$\frac{\partial \phi}{\partial t} = -g\eta,$$

where $\phi$ is the velocity potential and $g$ is the acceleration due to gravity.

(a) State the partial differential equation governing $\phi$ in the interior of the fluid.

(b) By considering the motion of a particle on the free surface show that the other linearised boundary condition on $\phi$ at $y = 0$ is

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t}.$$

(c) State, and briefly justify, a boundary condition on $\phi$ as $y \to -\infty$.

(d) If $\eta(x,t) = \epsilon \sin[(2\pi/\lambda)(x - ct)]$ with $\epsilon \ll \lambda$, show that the velocity potential is

$$\phi = -\epsilon c \exp(2\pi y/\lambda) \cos[(2\pi/\lambda)(x - ct)],$$

where the wavespeed, $c$, is given by the dispersion relation

$$c^2 = \lambda g/2\pi.$$

(e) Show that (to leading order in $\epsilon$) the particle paths are circular and sketch these paths for particles at various depths.

6. The depth of water in a channel is equal to $h$ at a location where the width is $d$. If the volume rate of flow is equal to $Q$, show that

$$\frac{Q^2}{2gd^2} = h^2(H - h),$$

where $H$ is a constant and $g$ is gravity. Show that, for a range of values of $Q$, there are two possible values of $h$, the larger of which lies between $\frac{2}{3}H$ and $H$.

If, for a fixed flow rate $Q$, the width of the channel increases by a small amount, and if $h$ lies in the range between $\frac{2}{3}H$ and $H$, does the depth of the water increase or decrease?