UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M351: Financial Mathematics

COURSE CODE	:	**MATHM351**
UNIT VALUE	:	**0.50**
DATE	:	**27-MAY-05**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

05-C0975-3-90

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

NOTE: In the questions which follow the current price of an asset (or similar instrument) will often be denoted either by S_t or simply by S with the time subscript suppressed. Reference may be made to the following definitions:

$$(x)^+ = \max\{x, 0\},$$

$$\Phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} \exp\{-\frac{z^2}{2}\}\, dz,$$

$$d_1 = \frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\ln(S/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

where K denotes the exercise price, r the riskless rate per unit time, σ^2 the volatility per unit time and T is the maturity date so that $T-t$ is the time to expiry.

The Black-Scholes formula for pricing a European call is:

$$S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2).$$

1. (a) In the context of a one-period multi-state model of asset prices define what is meant by *arbitrage opportunity* and *risk-neutral measure*. State and prove the No-Arbitrage Theorem.

(b) Consider the following model with $r = 0$, and two assets.

n	$S_n(0)$	$S_n(1,\omega_1)$	$S_n(1,\omega_2)$	$S_n(1,\omega_3)$
1	6	7	7	5
2	11	13	9	9

Show that there is no risk-neutral probability measure for this model. Find an arbitrage opportunity.

(c) An asset is currently priced at \$30. At the end of a year it will be worth either \$20 or \$40. If the risk-free annual interest rate is $r = 1/9$, what is the value of a European call option that expires in one year and has a strike price of \$30? [Treat this as a single-period model.]

2. (a) Consider the following model, with interest rate $r = 0$:

ω	$S(0)$	$S(1)$	$S(2)$
ω_1	9	15	17
ω_2	9	15	11
ω_3	9	7	11
ω_4	9	7	3

Replicate the call option $X = (S(2) - 7)^+$ over the two periods and so find the fair price of the claim at time $t = 0$.

(b) For the model in (a), find all the one-period risk-neutral probability measures, and the corresponding probability measure on $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. Determine the time 0 value of the Asian option

$$Y = \left(\frac{1}{3}[S(0) + S(1) + S(2)] - 9\right)^+.$$

(c) In the T-period binomial model of asset dynamics for a single asset if the asset price is S at any time, the next period's price will be either SU or SD. The interest rate per period r is positive and $D^* < 1 < U^*$, where the star denotes discounting.

(i) Describe the risk-neutral measure $\mathbb{Q}$.

(ii) Suppose that the price of the asset is S_0 at time $t = 0$. Using $\mathbb{Q}$ deduce that in this model the value of a European call with expiry at time $t = T$, written on the asset and having strike price K is either zero or (for some $\hat{n}$ which you should find) equal to

$$\frac{1}{(1+r)^T} \sum_{n=\hat{n}}^{T} \binom{T}{n} q^n (1-q)^{T-n} (U^n D^{T-n} - K),$$

where

$$q = \frac{1 + r - D}{U - D}$$

3. (a) Let Ω be a finite set, and let $\mathbb{P}$ be a probability measure on Ω. Define what is meant by a *filtration* $\{P_t : t = 0, ..., T\}$ on Ω. When is a process $S(t)$ said to be *adapted* to the filtration, and when is it a *martingale*? When is a process $H(t)$ *previsible* with respect to a filtration?

(b) Give a brief explanation of the idea behind dynamical programming as applied to the valuation of an American option. Use the method to value an American call option with exercise price $K = 7$ dollars written on an asset where the asset prices in dollars are given below, the interest rate per period is zero, and a dividend of two dollars is payable at time $t = 1.5$.

state	$t=0$	$t=1$	$t=2$
ω_1	9	15	15
ω_2	9	15	9
ω_3	9	7	9
ω_4	9	7	3

What is the optimal stopping time for this option? How should the option be hedged if it were to be sold at time 0 ?

4. (a) Let $f(S,t)$ be a function of two variables (continuously twice differentiable in S and once in t). State Itô's Formula for $df(S(t),t)$, where $S(t)$ is an asset price obeying the stochastic equation

$$dS = \mu dt + \sigma dW,$$

in which $W = W(t)$ is standard Brownian motion and μ, σ are continuous functions of S and t. Give a plausibility argument in support of the formula.

(b) Find an expression for

$$\int_0^T W dW.$$

Show that

$$\int_0^T W^2 dW = \frac{1}{3}(W(T)^3 - W(0)^3) - \int_0^T W dt.$$

[Hint: In the first case, use $S(t) = W(t)$ and $f(S) = \frac{1}{2}S^2$. For the second case, choose a different f.]

5. (a) Let $V(S,t)$ denote the value at time $t \leqslant T$ of a European option when the price of the underlying asset is S. Assume that the asset price process $S(t)$ follows the stochastic equation

$$dS = \mu S dt + \sigma S dW, \tag{1}$$

where $W = W(t)$ is a standard Brownian motion, μ, σ are constants and r is a constant riskless interest rate applicable throughout the life of the option.

Use Itô's Formula to derive the Black-Scholes equation satisfied by the function $V(S,t)$, namely

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} = rV.$$

(b) Show that the time-independent solutions of the equation in part (a) take the form $AS + BS^{-\gamma}$ for constants A, B for a suitably chosen positive γ which you should specify.

Suppose that $S \leq K$ and $r > 0$. A *perpetual dollar-or-nothing option* with exercise price K can be exercised at any time, paying one dollar if the price of the asset is at least K and nothing otherwise. Show that the value of such an option is S/K.

[Hint: You will need to consider boundary conditions as $S \to 0$ and at $S = K$. You should argue that the value tends to 0 as $S \to 0$, and also justify your choice of value at $S = K$.]