UNIVERSITY COLLEGE LONDON

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University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M351: Financial Mathematics

COURSE CODE : **MATHM351**

UNIT VALUE : **0.50**

DATE : **13-MAY-04**

TIME : **10.00**

TIME ALLOWED : **2 Hours**

04-C0972-3-50

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

NOTE: In the questions which follow the current price of an asset (or similar instrument) will often be denoted either by S_t or simply by S with the time subscript suppressed. Reference is made to the following definitions:

$$(x)^+ = \max\{x, 0\},$$

$$\Phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} \exp\{-\frac{z^2}{2}\}\, dz,$$

$$d_1 = \frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\ln(S/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

where K denotes the exercise price, r the riskless rate per unit time, σ^2 the volatility per unit time and T is the maturity date so that $T-t$ is the time to expiry.

The Black-Scholes formula for pricing a European call is:

$$S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2).$$

1. (a) In the context of a one-period multi-state model of asset prices define what is meant by 'arbitrage opportunity' and 'risk neutral measure'. State and prove the No-Arbitrage Theorem.

(b) Suppose the Government charges a citizen tax equal to a fixed proportion θ of the part of his annual income which is in excess of a 'tax threshold' of K.

A citizen's annual income is assumed to be directly proportional to a market index I evaluated at the end of the year (so that if the end of year index value is I he earns βI in that year). Explain why the citizen may be regarded by the Government as a call option on the index value next year with strike price K/β. In what sense does this observation provide the Government the opportunity to issue a guaranteed debt equal to the call value?

The index is currently I_0. In order to make High and Low projections regarding a citizien's tax, the Government makes projections about the next year's index using a two-state model and assumes the index value will be either Higher or Lower than currently with respective value I_H or I_L (so that $I_H > I_0(1+r) > I_L$, where r is the interest rate).

Find by how much they should discount their high tax projection to obtain a valuation of the citizen as a tax-revenue source. Remember to include discounting by r and to distinguish between the case where $\beta I_L < K$ and $\beta I_L > K$.

2. (a) Assume a zero interest rate and suppose the following (binomial) model is used to describe the price of a risky asset at times $t = 0,1,2$.

state	$t=0$	$t=1$	$t=2$
ω_1	4	8	16
ω_2	4	8	4
ω_3	4	2	4
ω_4	4	2	1

The call option $(S(2) - S(1))^+$ is payable at time $t = 2$ using as exercise price the asset price realised at time $t = 1$.Find the time $t = 0$ fair price of this option and determine how to replicate this claim. (This is an example of a 'forward-start, at the money' option.)

(b) In the T-period binomial model of asset dynamics for a single asset if the asset price is S at any time, the next period's price will be either SU or SD. The interest rate per period r is positive and $D^* < 1 < U^*$, where the star denotes discounting. Describe the risk-neutral measure Q.

(i) Suppose that the price of the asset at time $t = 1$ is observed to be S_1. Using Q deduce that in this model the value at time $t = 1$ of a European call with expiry at time $t = 1$, written on the asset and having strike price 'at the money', i.e. $K = S_1$, is for some $\hat{n}$ (which you should find) equal to

$$S_1 \sum_{n=\hat{n}}^{T'} \binom{T'}{n} \hat{q}^n (1-\hat{q})^{T'-n} - \frac{S_1}{(1+r)^{T'}} \sum_{n=\hat{n}}^{T'} \binom{T'}{n} q^n (1-q)^{T'-n}$$

where $T' = T - 1, q = \frac{1+r-D}{U-D}$ and $\hat{q} = qU/(1+r)$.

(ii) Suppose that at time $t = 0$ the asset price is S_0. Find the value at time $t = 0$ of the 'forward-start' call option paying at time T the claim $(S(T) - S(1))^+$, where the strike price is determined by the asset price of time $t = 1$, i.e. after this price has been observed.

Hint: Use the one-period measure and the formula in part (i).

3. Define what is meant by: (a) a *partition* P_t of a finite sample space Ω ('set of states') corresponding to a time $t = 0, 1, ..., T,$; (b) a *filtration* $\{P_t : t = 0, ..., T\}$. When is a process $S(t)$ said to be *adapted* to the filtration, and when is it a *martingale* with respect to a measure P on Ω? When is a process $H(t)$ *predictable* with respect to a filtration?

If it is known for each t with $0 \leqslant t < T$ that $E_P[S_n(t+1)|P_t] = S_n(t)$ deduce that $E_P[S_n(t+u)|P_t] = S_n(t)$ for each $t \geqslant 0$ and each $u > 0$ with $t + u \leqslant T$.

Briefly explain the idea behind dynamical programming as applied to the valuation of an American option. Use the method to value an American call option with exercise price $K = 5$ dollars written on an asset where the asset prices in dollars are given below, the interest rate per period is zero, and a dividend of one dollar is payable at time $t = 1.5$.

state	$t=0$	$t=1$	$t=2$
ω_1	7	10	10
ω_2	7	10	7
ω_3	7	6	7
ω_4	7	6	4

What is the optimal stopping time for this option? How should the option be hedged if it were to be sold at time 0 ?

4. (a) Let $f(S,t)$ be a function of two variables (continuously twice differentiable in S and once in t). State Itô's Formula for $df(S_t,t)$, where S_t is an asset price obeying the stochastic equation

$$dS_t = adt + bdz_t,$$

in which z_t is standard Brownian motion and a, b are continuous functions of S and t. Give a plausibility argument in support of the formula.

(b) You are told that the option pricing equation for the fair price u of an option in terms of dimensionless asset and time variables has the form

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + (\rho - 1)\frac{\partial u}{\partial x} - \rho u.$$

Here x is the logarithm of asset price, τ is the time to expiry (measured by the dimensionless time parameter) and ρ is the riskless rate per unit of dimensionless time. [In the notation of page 1: $x = \log(S/K), \tau = \frac{1}{2}\sigma^2(T-t)$, and $\rho = 2r/\sigma^2$.]

(i) Use the substitution

$$u(x,\tau) = e^{-\rho\tau}w(x+\beta\tau,\tau)$$

for an appropriate choice of β to reduce the option pricing equation to the format

$$\frac{\partial w}{\partial \tau} = \frac{\partial^2 w}{\partial x^2}.$$

(ii) Show that the equation in (i) has the solution $w = V(x/\sqrt{\tau})$ where $V(z)$ is a function of one variable and solves the equation

$$V'' + \frac{1}{2}zV'(z) = 0.$$

(iii) Solve the equation for V by integration and deduce that a solution of the option pricing equation is

$$u(x,\tau) = e^{-\rho\tau}[A\Phi([x+(\rho-1)\tau]/\sqrt{2\tau}) + B]$$

(Hint: V''/V' is the derivative of $\log V'(z)$.) Interpret this answer: what does it say about the dollar-or-nothing option?

5. (a) Let $V(S,t)$ denote the value at time $t \leqslant T$ of a European option when the price of the underlying asset is S. Assume that the asset price process S_t follows the stochastic equation

$$dS_t = \mu S_t dt + \sigma S_t dz_t, \tag{1}$$

where z_t is a standard Brownian motion, μ, σ are constants and r is a constant riskless interest rate applicable throughout the life of the option.

Use Itô's Formula to derive the Black-Scholes equation satisfied by the function $V(S,t)$, namely

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} = rV.$$

(b) Show that the time independent solutions of the equation in part (a) take the form $AS + BS^{-\gamma}$ for constants A, B for a suitably chosen positive γ which you should specify.

(c) A firm has assets whose value S_t at any time t obeys the equation (1). The firm is legally prevented from continuing its business as soon as the value of its assets falls to a pre-determined level S_B (when bankruptcy is said to occur).

The firm has contracted with its bank to make a cash payment continuously at a rate of C per unit time. In the absence of bankruptcy the present value of the contract to the bank is worth C/r where r is the assumed constant riskless deposit rate. If bankruptcy occurs the assets are sold at a cost αS_b where $0 < \alpha < 1$ and the bank receives the residue $(1-\alpha)S_B$. The possibility that the firm may fail thus reduces the value of the contract below C/r to a value $D(S)$ dependent on the current value S of firm assets.

Assume that $W = \frac{C}{r} - D$ satisfies

$$\frac{1}{2}\sigma^2 S^2 \frac{d^2 W}{dS^2} + rS\frac{dW}{dS} - rW = 0$$

and use part (b) to find the value of debt. (Hint: $D(S) \to C/r$ as $S \to \infty$.) Deduce that the risk-neutral probability of bankruptcy is

$$q_B = (S/S_B)^{-\rho}.$$