UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sc.* *M.Sci.*

Mathematics M351: Financial Mathematics

COURSE CODE	:	**MATHM351**
UNIT VALUE	:	**0.50**
DATE	:	**30-MAY-03**
TIME	:	**10.00**
TIME ALLOWED	:	**2 Hours**

03-C0963-3-60

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is not *permitted in this examination.*

NOTE: In the questions which follow the current price of an asset (or similar instrument) will often be denoted either by S_t or simply by S with the time subscript suppressed. Reference is made to the following definitions:

$$(x)^+ = \max\{x, 0\},$$

$$\Phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} \exp\{-\frac{z^2}{2}\}\, dz,$$

$$d_1 = \frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln(S/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

The Black-Scholes formula for pricing a European call is:

$$S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2).$$

1. (a) In the context of a one-period multi-state model of asset prices define what is meant by 'arbitrage opportunity' and 'risk neutral measure'. State and prove the No-Arbitrage Theorem.

   (b) A certain bank offers a deposit scheme with maturity at time $t = T$ whereby for each pound deposited (at time $t = 0$) the bank either returns the money (without interest) or else pays an amount $\alpha I(T)/I(0)$ pounds, where $I(t)$ denotes a specified market indicator value at the times $t = 0$ and $t = T$, and $\alpha < 1$. Show that this contract is equivalent to the bank offering its depositors a number of units of a call option on the market indicator struck at k times $I(0)$ with $k = 1/\alpha$, and charging them for it by not paying interest on the deposit.

   Assume now that $I(0) = 1$. Show how to select α so that the contract is fairly priced using a two-state one-period model in which $I(1)$ is either equal to I_H or I_L.

2. **Refer to the start of this examination paper for notation.**

(a) Assuming a zero interest rate and using the following model for the price of a risky asset at times $t = 0$ and $t = 1$

ω	$S(0,\omega)$	$S(1,\omega)$	$S(2,\omega)$
ω_1	5	9	17
ω_2	5	9	5
ω_3	5	3	5
ω_4	5	3	2

find the time $t = 0$ fair price of the put option $(4 - S(2))^+$.

(b) In the T-period binomial model of asset dynamics for a single asset, if the asset price is S at any time, the next period's price will be either SU or SD. The initial price of the asset is unity and the interest rate per period R is positive and $D^* < 1 < U^*$, where the star denotes discounting. Describe the risk-neutral measure Q.

A cash-or-nothing option written on the asset just described pays one dollar at time $t = T$ when that asset price is at or above a fixed level K and is worthless otherwise. Using Q, show that the option value at time $t = 0$ is equal to

$$\frac{1}{(1+R)^T} \sum_{n=\widehat{m}}^{T} \binom{T}{n} q_u^n q_d^{T-n}$$

for some $\widehat{m}$ (which you may need to identify).

Show that if $1 + R = e^{r\Delta t}$ and $U = e^{\sigma\sqrt{\Delta t}}$, $D = e^{-\sigma\sqrt{\Delta t}}$ where r and σ are positive constants and Δt is a positive variable, then

$$\lim_{\Delta t \to 0} (1 - 2q)/\sqrt{\Delta t} = -\left(\frac{r}{\sigma} - \frac{1}{2}\sigma\right).$$

Explain briefly the connection between this limiting formula and the value $\Phi(d_2)$.

3. Define what is meant by

(a) a *partition* P_t of a finite sample space Ω ('set of states') corresponding to a time $t = 0, 1, ..., T$,

(b) a *filtration* $\{P_t : t = 0, ..., T\}$.

When is a process S_t said to be *adapted* to the filtration, and when is it a *martingale* with respect to a measure P on Ω? When is a process H_t *predictable* with respect to a filtration? What is an *arbitrage opportunity* for a price process $\{S_n(t) : n = 0, 1, ...N\}$ adapted to the filtration.

Show that there is no such arbitrage opportunity if and only if all the one period submodels have no arbitrage oppportunities. (You may refer to the multi-period and one-period No Arbitrage theorems.)

Use the dynamical programming method to value an American call option with exercise price $K = 5$ written on an asset where the asset prices in dollars are given below, the interest rate per period is zero, and a dividend of one dollar is payable at time $t = 1.5$.

ω	$S(0,\omega)$	$S(1,\omega)$	$S(2,\omega)$
ω_1	6	9	9
ω_2	6	9	6
ω_3	6	5	6
ω_4	6	5	3

How should the option be hedged if it were to be sold at time 0 ?

4. Let $f(S,t)$ be a function of two variables (continuously twice differentiable in S and once in t). State Itô's Formula for $df(S_t,t)$, where S_t is an asset price obeying the stochastic equation

$$dS_t = adt + bdz_t,$$

in which z_t is standard Brownian motion and a, b are continuous functions of S and t. Give a plausibility argument in support of the formula.

Use the formula to show that

$$\int_0^T z^2(t)dz(t) = \frac{1}{3}(z(T)^3 - z(0)^3) - \int_0^T z(t)dt.$$

What is the mean and variance of the risk-neutral probability of the asset price S_T given its price S_t at time t when $a = \mu S$ and $b = \sigma S$ with μ, σ constant?

A 'power option' on the asset with price S_t as just described, has a strike price K and matures at time T, at which time it pays S_T^{-2} to the holder if and only if $S_T \geqslant K$ but is worthless otherwise. Find the option value at time $t = 0$ using the risk-neutral probability density. You should present your answer in terms of the cumulative normal distribution function $\Phi(u)$ defined at the start of this examination paper.

5. (a) Let $V(S,t)$ denote the value at time $t \leqslant T$ of a European option when the price of the underlying asset is S. Assume that the asset price process S_t follows the stochastic equation

$$dS_t = \mu S_t dt + \sigma S_t dz_t,$$

where z_t is a standard Brownian motion, μ, σ are constants and r is a constant riskless interest rate applicable throughout the life of the option. Assume further that the asset pays a dividend continuously at rate q of value $qS_t \Delta t$ in a period of length Δt.

Use Itô's Formula to derive the Black-Scholes equation satisfied by the function $V(S,t)$, namely

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} = rV.$$

(b) Show that the time independent solutions of the equation in part (a) take the form $AS^{\alpha} + BS^{-\beta}$ for constants A, B for suitably chosen positive α and β which you should specify.

(c) A perpetual barrier option with a fixed rebate-rate k pays kJ_t at the first time that the asset value S_t falls to exactly kJ_t where J_t is the maximum value of the asset price up to time t. Assume the option value is time-independent and takes the form $V(S_t, J_t)$. Find this value using part (b) by writing $V(S,J) = JW(S/J)$, where $W(x)$ is a function of one variable. You should assume that

$$\frac{\partial V}{\partial J}(S,S) = 0$$

(i.e. the option value is insensitive to small changes in the current maximum as that is unlikely to remain the maximum later). You may find it useful to verify that $W(k) = k$ and $W(1) = W'(1)$.