UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. *M.Sc.* *M.Sci.*

Mathematics M351: Financial Mathematics

COURSE CODE	:	**MATHM351**
UNIT VALUE	:	**0.50**
DATE	:	**16-MAY-02**
TIME	:	**14.30**
TIME ALLOWED	:	**2 hours**

02-C0958-3-50

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

NOTE: In the questions which follow the current price of an asset (or similar instrument) will often be denoted either by S_t or simply by S with the time subscript suppressed. Reference is made to the following definitions:

$$(x)^+ = \max\{x, 0\},$$

$$\Phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} \exp\{-\frac{z^2}{2}\}\, dz,$$

$$d_1 = \frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln(S/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

The Black-Scholes formula for pricing a European call is:

$$S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2).$$

1. (a) In the context of a one-period multi-state model of asset prices define what is meant by 'arbitrage opportunity' and 'risk neutral measure'. State and prove the No-Arbitrage Theorem.

(b) The price of an asset is initially unity, i.e. $S(0) = 1$. The price at time 1 may be one of three values u, v, w, where $0 < u < v < w$. Write down all the risk neutral measures when the interest rate r satisfies $u < 1 + r < v$ and when it satisfies $v < 1 + r < w$. What happens when $1 + r > w$?

(c) A certain car insurance, which pays any level of claim in full, is sold for £500. An alternative policy is sold for £40 less with the restriction that the client meets the first £200 of any claim.

(i) Examine this market for arbitrage opportunities when modelling the sample space of claims with two states: no claim and a single possible claim at £5000.

(ii) What happens if the sample space is extended to contain a further possible claim of £2000.

2. (a) Assuming a zero interest rate and using the following model for the price of a risky asset at times $t = 0$ and $t = 1$

ω	$S(0,\omega)$	$S(1,\omega)$	$S(2,\omega)$
ω_1	6	9	10
ω_2	6	9	7
ω_3	6	5	7
ω_4	6	5	4

find the value of the following look-back option

$$X(\omega) = \max\{S(t,\omega) - 8 : t = 0, 1, 2\}.$$

(b) In the T-period binomial model of asset dynamics for a single asset, if the asset price is S at any time, the next period's price will be either SU or SD. The initial price of the asset is unity, the interest rate per period r is positive and $D^* < 1 < U^*$, (where the star denotes discounting). Describe the risk-neutral measure Q.

(i) Using Q find the value P of a European *put* written on the asset and having strike price K.

(ii) Recall that the value of a European *call* with strike price K either is zero or, for some $\widehat{m}$ (which you may need to identify) is equal to

$$C = \sum_{n=\widehat{m}}^{T} \binom{T}{n} \hat{q}^n (1-\hat{q})^{T-n} - \frac{K}{(1+r)^T} \sum_{n=\widehat{m}}^{T} \binom{T}{n} q^n (1-q)^{T-n},$$

where $q = \frac{1+r-D}{U-D}$, $\hat{q} = qU/(1+r)$.

Using your formula for P from (i), verify the parity relation

$$P - C = K^* - 1.$$

3. Define what is meant by

(a) a *partition* P_t of a finite sample space Ω ('set of states') corresponding to a time $t = 0, 1, ..., T$,

(b) a *filtration* $\{P_t : t = 0, ..., T\}$.

When is a process S_t said to be *adapted* to the filtration, and when is it a *martingale* with respect to a measure P on Ω?

If it is known for each t with $0 \leqslant t < T$ that $E_P[S_n(t+1)|P_t] = S_n(t)$, deduce that $E_P[S_n(t+u)|P_t] = S_n(t)$ for each $t \geqslant 0$ and each $u > 0$ with $t + u \leqslant T$.

Briefly explain the idea behind dynamical programming as applied to the valuation of an American option. Use the method to value an American *call* option with exercise price $K = 5$ written on an asset where the asset prices in dollars are given below, the interest rate per period is zero, and a dividend of one dollar is payable at time $t = 1.5$.

ω	$S(0,\omega)$	$S(1,\omega)$	$S(2,\omega)$
ω_1	5	8	9
ω_2	5	8	6
ω_3	5	4	6
ω_4	5	4	2

Show that it is sometimes optimal to exercise at time $t = 1$.

How should the option be hedged if it were to be sold at time 0.

4. Let $f(S,t)$ be a function of two variables (continuously twice differentiable in S and once in t). State Itô's Formula for $df(S_t,t)$, where S_t is an asset price obeying the stochastic equation

$$dS_t = adt + bdz_t,$$

in which z_t is standard Brownian motion and a,b are continuous functions of S and t. Give a plausibility argument in support of the formula.

Use the formula to show that

$$\int_0^T z_t dz_t = \frac{1}{2}(z_T^2 - z_0^2) - \frac{1}{2}T.$$

What are the mean and variance of the risk-neutral probability of the asset price S_T at time T when $a = \mu S$ and $b = \sigma S$ with μ, σ constant?

Use this probability to evaluate the fair price at time $t < T$ of a *put* option on the asset with exercise price K and maturing at time T, by expressing the conditional expectation $E_t[(K - S_T)^+]$ suitably discounted in terms of the cumulative normal distribution function $\Phi(u)$ defined at the start of this examination paper.

5. (a) Let $V(S,t)$ denote the value at time $t \leqslant T$ of a European option when the price of the underlying asset is S. Assume that the asset price process S_t follows the stochastic equation

$$dS_t = \mu S_t dt + \sigma S_t dz_t,$$

where z_t is a standard Brownian motion, μ, σ are constants and r is a constant riskless interest rate applicable throughout the life of the option. Assume further that the asset pays a dividend continuously at rate q of value $qS_t\Delta t$ in a period of length Δt.

Use Itô's Formula to derive the Black-Scholes equation satisfied by the function $V(S,t)$, namely

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} = rV.$$

(b) Show that the time independent solutions of the equation in part (a) take the form $AS^\alpha + BS^{-\beta}$ with A, B constant, for some appropriate positive α and β which you should specify.

Use this result to deduce that when $q \geqslant 0$ a perpetual dollar-or-nothing option with exercise price K has value $(S/K)^\alpha$ for $S \leqslant K$.