UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M234: Electricity and Magnetism

COURSE CODE	:	**MATHM234**
UNIT VALUE	:	**0.50**
DATE	:	**07-MAY-03**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

03-C0954-3-50

TURN OVER

All questions may be attempted but only marks obtained on the best ***four*** *solutions will count.*
The use of an electronic calculator is permitted in this examination.

1. (a) State the vacuum versions of Maxwell's equations and the Lorentz force law. Which term in Maxwell's equations was due to Maxwell himself? What is the name of this term?

   (b) In each case starting from the vacuum versions of Maxwell's equations:

   (i) Show that
$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0,$$
   and explain what the equation represents.

   (ii) Show, if $\rho = 0$ and $\mathbf{J} = \mathbf{0}$, that
$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \mathbf{0},$$
   and identify the constant c and give its physical interpretation.

   (iii) Show, without using $\nabla \cdot \mathbf{B} = 0$, that
$$\frac{\partial}{\partial t}(\nabla \cdot \mathbf{B}) = 0.$$

2. (a) For a vacuum, define the Poynting vector $\mathbf{S}$ and the electromagnetic energy density U. What does $\mathbf{S}$ represent physically?

   (b) Derive
$$\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{S} + \mathbf{E} \cdot \mathbf{J} = 0.$$

   (c) Determine U and $\mathbf{S}$ for an electromagnetic plane wave in a source-less vacuum and show that they satisfy the equation of part 2b. Define any additional symbols that you introduce.

3. (a) State the electromagnetic media form of Maxwell's equations in differential form, giving the definitions of the fields $\mathbf{D}$ and $\mathbf{H}$. What are the physical interpretations of the polarization field $\mathbf{P}$ and magnetization field $\mathbf{M}$?

   (b) Determine the fields $\mathbf{E}$ and $\mathbf{D}$ everywhere for a system consisting of a uniformly polarized ball of radius a with constant polarization $\mathbf{P}_0$.

4. Throughout this question, the vacuum versions of Maxwell's equations are assumed.

   (a) Determine the electrostatic energy U_e in a parallel plate capacitor of plate area A and plate separation d when the plates have equal and opposite charges of magnitude Q. State any standard approximations used clearly. Sketch the physical system.

   (b) Determine the magnetostatic energy U_m in a long thin circular cross-sectional solenoid of length ℓ and radius a with n turns per unit length, when the wire is carrying a current I. State any standard approximations used clearly. Sketch the physical system.

   (c) Assuming that the solutions for parts 4a and 4b are approximately valid for the time-dependent case, and that each end of the wire from the solenoid is connected to a different plate of the capacitor, show that this system supports a sinusoidal oscillation and determine its frequency. You may assume that energy is conserved, but any other assumptions should be clearly stated. Where might such a tuned circuit be found in your home?

5. (a) The mean magnetic flux density within the Earth is defined by

$$\bar{\mathbf{B}} = \frac{1}{V}\int_V \mathbf{B}dV,$$

   where V is the interior of the Earth. Using the divergence theorem or otherwise, show that $\bar{\mathbf{B}}$ can be expressed as a surface integral involving $\mathbf{B}$. Why might the surface-integral form of the result be more useful in practice?

   Hint: You might find it helpful to consider the expansion of $\nabla \cdot ((\boldsymbol{B})(\boldsymbol{x} \cdot \mathbf{c}))$, *where* $\mathbf{c}$ *is a constant vector.*

   (b) Suppose that all space is occupied by a conductor at rest, with conductivity σ, which is constant and uniform. If the magnetic flux density in Cartesian coordinates is of the form

$$\mathbf{B}(x, y, z, t) = (0, 0, B(x, y, t)),$$

   find the evolution equation for B.

6. Consider a system consisting of a steady current I flowing through a thin wire loop C. The vector potential for a magnetic flux density $\mathbf{B}$ for this system is

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0 I}{4\pi} \oint_C \frac{d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}.$$

(a) By using Stokes' theorem, or otherwise, show that

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0 I}{4\pi} \int_S \frac{\mathbf{n}(\mathbf{y}) \times (\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3} dS_{\mathbf{y}},$$

where S is a surface with boundary C and unit normal field $\mathbf{n}$.

(b) What is meant by the far-field approximation of $\mathbf{A}(\mathbf{x})$? Show that this approximation leads to

$$\mathbf{A}(\mathbf{x}) \sim \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times (\mathbf{x}-\mathbf{y}_0)}{|\mathbf{x}-\mathbf{y}_0|^3},$$

where $\mathbf{m}$ and $\mathbf{y}_0$ are to be defined. What is the name given to $\mathbf{m}$?

(c) Show that, if C is on a plane with unit normal $\mathbf{n}$, then $\mathbf{m}$ in 6b of this question becomes

$$\mathbf{m} = I\mathbf{n}|S|,$$

where $|S|$ is the area of the plane within C.