UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Dynamical Systems

COURSE CODE : **MATH3509**

UNIT VALUE : **0.50**

DATE : **05-MAY-06**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

06-C0990-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. a) State conditions which ensure that a map f is chaotic on a set S. Explain all terms you use.

   b) Consider the discrete dynamical system on the interval $[-1, 1]$ defined by

$$x_{n+1} = f(x_n)$$

   where $f(x) = 4x^3 - 3x$. Calculate the fixed points and critical points of $f(x)$ in $[-1, 1]$. Calculate the values of f at the critical points, and at $x = 1$ and $x = -1$.

   c) Let $h : \mathbb{S}^1 \to [-1, 1]$ be the map defined by $h(\theta) = \cos(\theta)$ and $g : \mathbb{S}^1 \to \mathbb{S}^1$ be the circle map defined by $g(\theta) = 3\theta$. Show that $h \circ g = f \circ h$. Assuming that g is chaotic on the whole circle, what does this imply about the map f? What would you additionally need to show to complete the proof that f is chaotic on $[-1, 1]$?

2. Consider a real map f defined on an interval $I \subset \mathbb{R}$.

   a) Given a closed interval $A \subset I$, prove that if $f(A) \supset A$, then f must have at least one fixed point in A.

   b) Assume that I contains two closed subintervals, A and B, and that $A \cup B$ contains an invariant subset S. Define the "itinerary" of a point in S. Explain what it means for a point to have a periodic itinerary.

   c) Assume that A and B satisfy $f(A) \supset A \cup B$ and $f(B) \supset A \cup B$. Describe the implications of this fact and explain briefly how you would prove your statements (a complete proof is not needed).

3. Consider the following ODE system on $\mathbb{R}^2$:

$$\begin{aligned}\dot{x} &= f(x,y) = -x + x^2 - 2xy \\ \dot{y} &= g(x,y) = -2y - 5xy + y^2\end{aligned}$$

a) Explain what it means for a fixed point of a flow to be asymptotically stable. Prove that the origin is an asymptotically stable fixed point of this system.

b) Show that the function $V(x,y) = \frac{1}{2}(x^2 + y^2)$ is a Liapunov function for the fixed point at 0. Show that $\dot{V}$ is strictly negative away from the origin in a wedge-shaped region of $\mathbb{R}^2$. Write down the inequalities which define this wedge-shaped region, and sketch the region.

4. Consider the following ODE system on $\mathbb{R}^2$:

$$\begin{aligned}\dot{x} &= f(x,y) = x \\ \dot{y} &= g(x,y) = -y + x^2\end{aligned}$$

a) Show that the system has a unique fixed point at the origin. Decide by linearisation whether this fixed point is stable or unstable.

b) Define an invariant set for a flow. Show that the equation $h(x,y) = y - x^2/3 = 0$ defines an invariant curve for this system.

c) Consider the change of co-ordinates from (x,y) to (u,v) defined by

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x \\ y - x^2/3 \end{bmatrix}$$

Write down the system in the new co-ordinates (u,v). Briefly describe the dynamics of the system in this co-ordinate system.

5. Consider the following ODE on $\mathbb{R}$:

$$\dot{x} = G(x, \mu)$$

μ is a parameter which can take any real value.

a) Write down all conditions on the function G which will ensure that the system undergoes a saddle-node bifurcation at some point $(\overline{x}, \overline{\mu})$.

b) Now consider the particular case $G(x, \mu) = x^3 - 2x^2 - \mu x + 2\mu$. Show that G has a fixed root at some $x = \alpha$ for all values of μ. Find the value of α and use this information to factor G.

c) Use your previous results to identify a value of (x, μ) where a saddle-node bifurcation appears to take place. Confirm that it does take place by checking all bifurcation conditions.