Question A1

Need to prove that

$$\left(P \wedge \left((P \wedge R) \vee \left(R \rightarrow (P \rightarrow Q)\right)\right)\right) \leftrightarrow P$$

is a tautology.

Let $W \equiv \left(P \wedge \left((P \wedge R) \vee \left(R \rightarrow (P \rightarrow Q)\right)\right)\right) \leftrightarrow P$.

$$
\begin{aligned}
W(P/T) &\Leftrightarrow \left(\mathrm{T} \wedge \left((\mathrm{T} \wedge R) \vee \left(R \rightarrow (\mathrm{T} \rightarrow Q)\right)\right)\right) \leftrightarrow \mathrm{T} \\
&\Leftrightarrow \left(R \vee \left(R \rightarrow (\mathrm{T} \rightarrow Q)\right)\right) \leftrightarrow \mathrm{T} && \text{identity law (twice)} \\
&\Leftrightarrow \left(R \vee \left(R \rightarrow (\mathrm{F} \vee Q)\right)\right) \leftrightarrow \mathrm{T} && \text{implication law} \\
&\Leftrightarrow \left(R \vee (R \rightarrow Q)\right) \leftrightarrow \mathrm{T} && \text{identity law} \\
&\Leftrightarrow \left(R \vee (\neg R \vee Q)\right) \leftrightarrow \mathrm{T} && \text{implication law} \\
&\Leftrightarrow \left((R \vee \neg R) \vee Q\right) \leftrightarrow \mathrm{T} && \text{associative law} \\
&\Leftrightarrow (\mathrm{T} \vee Q) \leftrightarrow \mathrm{T} && \text{excluded middle law} \\
&\Leftrightarrow \mathrm{T} \leftrightarrow \mathrm{T} && \text{domination law} \\
&\Leftrightarrow \mathrm{T} && \text{truth tables}
\end{aligned}
$$

$$
\begin{aligned}
W(P/F) &\Leftrightarrow \left(F \wedge \left((F \wedge R) \vee \left(R \rightarrow (F \rightarrow Q)\right)\right)\right) \leftrightarrow F \\
&\Leftrightarrow F \leftrightarrow F && \text{domination law} \\
&\Leftrightarrow T && \text{truth tables}
\end{aligned}
$$

Since $\mathrm{W}(P/T)$ and $\mathrm{W}(P/F)$ are tautologies then so is W.

Question A2

a) $\forall x \cdot \left(P(x) \to Q(x) \right)$

Model:

$D = \{1\}$
$P(x): \quad x \neq 1$
$Q(x): \quad x \neq 1$

Both P(1) and Q(1) are true and hence a model for $P(1) \to Q(1)$. Since 1 is the only element of D then $P(x) \to Q(x)$ is true for <u>all</u> elements of D.

Countermodel:

Seek a model for
$\neg \forall x \cdot \left(P(x) \to Q(x) \right)$
$\Leftrightarrow \quad \exists x \cdot \neg \left(P(x) \to Q(x) \right)$
$\Leftrightarrow \quad \exists x \cdot \left(P(x) \wedge \neg Q(x) \right)$

$D = \{1\}$
$P(x): \quad x = 1$
$Q(x): \quad x \neq 1$

P(1) is true and Q(1) is false. Hence $P(1) \wedge \neg Q(1)$ is true and therefore there <u>exists</u> an element of D (1), such that $P(x) \wedge \neg Q(x)$ is true.

b)

$\forall x \cdot \forall y \cdot \left(P(x,y) \rightarrow \neg P(x,y)\right)$

$\Leftrightarrow \quad \forall x \cdot \forall y \cdot \left(\neg P(x,y) \vee \neg P(x,y)\right)$

$\Leftrightarrow \quad \forall x \cdot \forall y \cdot \neg P(x,y)$

Model:

$D = \{1\}$

$P(x,y): \qquad x \neq y$

$\neg P(1,1)$ is true since $P(1,1)$ is false. Since the 1 is the only element of D, it is the the case that for <u>all pairs</u> of elements in D we have $\neg P(x,y)$ being true. Hence a model for $\forall x \cdot \forall y \cdot \left(\neg P(x,y)\right)$.

Countermodel:

Seek a model for $\neg \forall x \cdot \forall y \cdot \neg P(x,y) \Leftrightarrow \exists x \cdot \exists y \cdot P(x,y)$.

$D = \{1\}$

$P(x,y): \qquad x = y$

We have a model for $P(1,1)$ since 1 = 1, and hence there exists a pair of elements in D such that $P(x,y)$ being true. Hence a model for $\exists x \cdot \exists y \cdot P(x,y)$.

Question A3

a) Let the digits 0,1,2....,9 represent the pigeonholes and the 91 integers represent the letters. Each integer is pigeonholed by its last digit. By the generalised pigeonhole principle, there exists a pigeonhole that contains at least $\lceil 91/10 \rceil = 10$ objects. Hence at least ten of the integers share the same last digit.

b) Assume that there is no strictly increasing or decreasing subsequence of $n+1$. Hence, for all k we have $1 \le i_k \le n,\ \ 1 \le d_k \le n$. Hence there are, at most, n^2 different pairings of (i_k, d_k) for the different k's. Since there are n^2+1 different ordered pairs then, by the pigeonhole principle, at least two of the elements of the sequence have the same pairing (i_k, d_k). Let these distinct elements be denoted by a_r, a_s and hence $(i_r, d_r) = (i_s, d_s)$ with $r < s$.

If $a_r < a_s$ then it must be the case that $i_r > i_s$ which leads to a contradiction.
If $a_r > a_s$ then it must be the case that $d_r > d_s$ which leads to a contradiction.

Hence a contradiction is arrived at and the original assumption is false. Hence there exists a strictly increasing or decreasing subsequence of length $n+1$.

Question A4

(a)

Assume that we have a group table that is not a Latin Square. Then one of two things must occur. Either:

(i) there is an element in the group table that does not exist in the group.

(ii) one of the rows/columns of the group table contains the same element twice.

Condition (i) cannot occur due to the closure property of groups.

Assume that condition (ii) occurs and that one of the rows of the group table contains the same element twice. Let that row correspond to element *a*, and therefore we have, for certain distinct *b* and *c*:

$$a \circ b = a \circ c$$

By the left cancellation rule, we thus have: $b = c$, which leads to a contradiction and thus condition (ii) cannot occur.

A very similar argument applies to a column containing the same element twice and, applying the right cancellation rule, we also arrive at a contradiction.

Since conditions (i) and (ii) cannot occur, a group table must be a latin square.

(b)

Consider a group of order three $\langle \{i, a, b\}, \circ \rangle$. From an empty group table we construct the options for the table as follows:

$$\begin{array}{c|ccc} & e & a & b \\ \hline e & & & \\ a & & & \\ b & & & \end{array} \Rightarrow \begin{array}{c|ccc} & e & a & b \\ \hline e & e & a & b \\ a & a & & \\ b & b & & \end{array} \Rightarrow \begin{array}{c|ccc} & e & a & b \\ \hline e & e & a & b \\ a & a & b & \\ b & b & e & \end{array} \Rightarrow \begin{array}{c|ccc} & e & a & b \\ \hline e & e & a & b \\ a & a & b & e \\ b & b & e & a \end{array}$$

Step 1 was due to the identity element.
Step 2 was due to our only choice in completing column 1.
Step 3 was due to our only choice in completing column 2.

There was only one choice in constructing this group.

Question A5

a) The matrix is symmetric and all its diagonal elements are ones. Hence the relation is symmetric and reflexive.

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \odot \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow \quad R \circ R \subseteq R$

$\Rightarrow \quad R$ is transitive.

Hence R is an equivalence relation.

b) Equivalence classes are $\{\{a,c,d\},\{b,e\}\}$.

c) Consider the relation given by the following matrix $\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$. In this relation we have symmetry and reflexive. However, $(a,b) \in R \wedge (b,d) \in R$ but $(a,d) \notin R$. Hence not transitive and hence R is not an equivalence relation.

Question A6

a)

i) No. In introducing a 1 into a string, it must be sandwiched between elements. Since the empty string is not used in the recursive definition, a 1 cannot be the first character in an element of *B*.

ii) Yes, $0 \in B \quad \Rightarrow \quad S_2(0,0) = 010 \in B \quad \Rightarrow \quad S_1(0,010) = 0010 \in B$

b) $0 \in B \quad \Rightarrow \quad S_1(0,0) = 00 \in B \quad \Rightarrow \quad S_2(00,0) = 0010 \in B$

Compare this with the result of a)ii) and we've generated the same element using two different approaches. Hence the set is not freely generated by the recursive definition.

c) $P(w): \; z(w) > o(w)$

Basis step: $P(0): \; z(0) > o(0) \quad \Leftrightarrow \quad 1 > 0 \quad \Leftrightarrow \quad \text{True}$

Hence basis step is satisfied.

Inductive step:

$$
\begin{aligned}
P(w) \wedge P(x) \quad &\Rightarrow \quad z(w) > o(w) \quad \wedge \quad z(x) > o(x) \\
&\Rightarrow \quad z(w) + z(x) > o(w) + o(x) \\
&\Rightarrow \quad z(S_1(w,x)) > o(S_1(w,x)) \\
&\Rightarrow \quad P(S_1(w,x))
\end{aligned}
$$

$$
\begin{aligned}
P(w) \wedge P(x) \quad &\Rightarrow \quad z(w) > o(w) \quad \wedge \quad z(x) > o(x) \\
&\Rightarrow \quad z(w) \geq o(w) + 1 \quad \wedge \quad z(x) \geq o(x) + 1 \\
&\Rightarrow \quad z(w) + z(x) \geq o(w) + o(x) + 2 \\
&\Rightarrow \quad z(w) + z(x) > o(w) + o(x) + 1 \\
&\Rightarrow \quad z(S_2(w,x)) > o(S_2(w,x)) \\
&\Rightarrow \quad P(S_2(w,x))
\end{aligned}
$$

Basis step and inductive step satisfied and hence proven $\forall w \cdot P(w)$.

Question B1

1.	$\neg(C \to D)$	Premise
2.	$C \to E$	Premise
3.	$E \to (G \vee D)$	Premise
4.	$\neg(\neg C \vee D)$	$1, \Leftrightarrow,$ implication law
5.	$\neg\neg C \wedge \neg D$	$4, \Leftrightarrow,$ deMorgans
6.	$C \wedge \neg D$	$5, \Leftrightarrow,$ double negation
7.	C	$6, \wedge\_E$
8.	$\neg D$	$6, \wedge\_E$
9.	E	$2, 7, \to\_E(MP)$
10.	$G \vee D$	$3, 9, \to\_E(MP)$
11.	G	$8, 10, \vee\_E$

b)
(i) To obtain a model for $\forall x \cdot P(x) \to \forall x \cdot Q(x)$ all we need is a countermodel for $\forall x \cdot P(x)$ which is a model for $\exists x \cdot \neg P(x)$.

To obtain a countermodel for $\forall x \cdot (P(x) \to Q(x))$ we need a model for $\neg \forall x \cdot (P(x) \to Q(x))$, that is, a model for $\exists x \cdot (P(x) \wedge \neg Q(x))$.

We therefore are looking for an interpretation that satisfies two existential quantifiers.

Example: Universe of discourse: $\{1, 2\}$
$P(x)$: "$x = 1$"
$Q(x)$: "$x = 2$"

Since $\neg P(2)$ is true we have a model for $\exists x \cdot \neg P(x)$.
Since $P(1) \wedge \neg Q(1)$ is true, we have a model for $\exists x \cdot (P(x) \wedge \neg Q(x))$.

Hence we have a model for the premise but a countermodel for the conclusion and therefore the argument is not valid.

(ii)

1.	$\forall x \cdot (P(x) \to Q(x))$	Premise
2.	$P(x) \to Q(x)$	$1, \forall\_E, x$ a free variable
3.	$\quad \forall x \cdot P(x)$	Assumption for $\to\_I$
4.	$\quad P(x)$	$3, \forall\_E$
5.	$\quad Q(x)$	$4, 2, \to\_E(MP)$
6.	$\quad \forall x \cdot Q(x)$	$5, \forall\_I$
7.	$\forall x \cdot P(x) \to \forall x \cdot Q(x)$	$3, 6, \to\_I$

Question B2

a) i)

$$\begin{aligned} S(n+1,r) &= S(n,r-1)+rS(n,r) \\ S(4,3) &= S(3,2)+3S(3,3) \\ &= S(3,2)+3 \\ &= S(2,1)+2S(2,2)+3 \\ &= S(2,1)+5 \\ &= 6 \end{aligned}$$

ii)

$$\{\{a,b\},\{c\},\{d\}\}$$
$$\{\{a,c\},\{b\},\{d\}\}$$
$$\{\{a,d\},\{b\},\{c\}\}$$
$$\{\{b,c\},\{a\},\{d\}\}$$
$$\{\{b,d\},\{a\},\{c\}\}$$
$$\{\{c,d\},\{a\},\{b\}\}$$

iii)

$$f=\{(a,1),(b,1),(c,2),(d,3)\}$$

(iii) The above surjection can be viewed as representing the partition $\{\{a,b\},\{c\},\{d\}\}$. This partition is the equivalence classes generated by the relation:

$$R=\{(x,y)\in X^2 \mid f(x)=f(y)\}$$

However, using this definition of *R*, a different surjection could have resulted in the same partition. Any surjection of the form $p\circ f : X\to W$ would give the same partition where *p* is any permutation of *W*. There are 3! different *p* functions and there are S(4,3) different *f* functions that generate unique partitions. Hence, by the product rule, there are 3!S(4,3) different surjections.

b)

(i) $\{\ (),\ (2\,3),\ (1\,6)(4\,5),\ (1\,6)(2\,3)(4\,5)\ \}$

(ii) $\{\{2,3\},\{1,6\},\{4,5\}\}$

(iii) Let the colours be represented by A, B, C and D. Let the colourings of the vertices be ordered so that, for example, the colouring ABCDBA represents vertices 1 and 6 being coloured A, vertices 2 and 5 being coloured B, and vertices 3 and 4 being coloured C and D, respectively.

Let X represent the set of all possible colourings.

$X = \{AAAAAA, ..., BBBBBB, ..., CCCCCC, ..., DDDDDD\}$
$|X| = 4^6$

We have, by the product rule:
$\left|fix(())\right| = 4^6$
$\left|fix((23))\right| = 4 \times 4^4$
$\left|fix((16)(45))\right| = 4 \times 4 \times 4^2$
$\left|fix((16)(23)(45))\right| = 4 \times 4 \times 4$

Hence, number of possible colourings is:

$$\frac{1}{4}\left(4^6 + 4^5 + 4^4 + 4^3\right) = \frac{4^3}{4}\left(64 + 16 + 4 + 1\right) = 16 \times 85 = 1360$$

Question B3

a)

$a_n - 4a_{n-1} + 3a_{n-2} = 4$

Homogeneous case

$h_n - 4h_{n-1} + 3h_{n-2} = 0$

$\Rightarrow \quad r^2 - 4r + 3 = 0$

$\Rightarrow \quad (r-3)(r-1) = 0$

$\Rightarrow \quad r = 1,\ 3$

$\Rightarrow \quad h_n = A\left(1^n\right) + B\left(3^n\right)$

$\Rightarrow \quad h_n = A + B\left(3^n\right)$

Inhomogeneous case

$t = 0, s = 1, m = 1$

$p_n = nC1^n = nC$

Particular solution satisfies the recurrence relation:

$p_n - 4p_{n-1} + 3p_{n-2} = 4$

$\Rightarrow \quad nC - 4(n-1)C + 3(n-2)C = 4$

$\Rightarrow \quad C = -2$

$\Rightarrow \quad p_n = -2n$

$a_n = A + B\left(3^n\right) - 2n$

$a_0 = 1 \quad \Rightarrow \quad 1 = A + B$

$a_1 = 3 \quad \Rightarrow \quad 3 = A + 3B - 2 \quad \Rightarrow \quad 5 = A + 3B$

$\Rightarrow \quad 4 = 2B \quad \Rightarrow \quad (A,B) = (-1,2)$

Hence:

$a_n = -1 + 2\left(3^n\right) - 2n$

b)

i)

$S_1 = \{1,2,3,4\}$

$S_2 = \{00,11,12,13,14,21,22,23,24,31,32,33,34,41,42,43,44\}$

ii) We are given in the question that the recurrence relation is first order and hence we expect a relation (possibly inhomogeneous) for a_n in terms of a_{n-1}. We aim to define the set S_n in a freely generated manner.

Let $x_{n-1} \in \Sigma_{n-1}$

1. If $x_{n-1} \in S_{n-1}$ then $a \in \{1,2,3,4\} \to (x_{n-1} \cdot a \in S_n)$
2. If $x_{n-1} \notin S_{n-1}$ then $x_{n-1} \cdot 0 \in S_n$

Hence we have $a_n = 4a_{n-1} + \tilde{a}_{n-1}$ where $\tilde{a}_{n-1} = |\Sigma_n - S_n|$.

We have $\tilde{a}_{n-1} = 5^{n-1} - a_{n-1}$ and hence $a_n = 3a_{n-1} + 5^{n-1}$

(*iii*) We have, from (*i*), $a_2 = 17$.

From (ii) we have: $a_2 = 3a_1 + 5^1 = 17$

Hence consistent.

Question B4

a) $P(n): \ 1+nh \leq (1+h)^n$

Basis step:

$$\begin{aligned} P(0): \ & 1+0h \leq (1+h)^0 \\ \Leftrightarrow \quad & 1 \leq 1 \\ \Leftrightarrow \quad & \text{True} \end{aligned}$$

Inductive step:

$$\begin{aligned} P(n): \ & 1+nh \leq (1+h)^n \\ \Rightarrow \quad & (1+nh)(1+h) \leq (1+h)^n(1+h) && \text{since } 1{+}h > 0 \\ \Rightarrow \quad & 1+nh+h+nh^2 \leq (1+h)^{n+1} \\ \Rightarrow \quad & 1+(n+1)h \leq (1+h)^{n+1} && \text{since } nh^2 > 0 \\ \Rightarrow \quad & P(n+1) \end{aligned}$$

Basis and inductive step satisfied and hence proven $\forall n \cdot P(n)$.

b)

$$\begin{aligned} a(1) &= 1-a(a(0)) = 1-a(0) = 1-0 = 1 \\ a(2) &= 2-a(a(1)) = 2-a(1) = 2-1 = 1 \\ a(3) &= 3-a(a(2)) = 3-a(1) = 3-1 = 2 \\ a(4) &= 4-a(a(3)) = 4-a(2) = 4-1 = 3 \\ a(5) &= 5-a(a(4)) = 5-a(3) = 5-2 = 3 \end{aligned}$$

c)
(i)

$$\begin{aligned} r(3,234) &= r(3,23\cdot 4) \\ &= r(3,23)\cdot 4 \\ &= r(3,2\cdot 3)\cdot 4 \\ &= r(3,2)\cdot 4 \\ &= r(3,\lambda\cdot 2)\cdot 4 \\ &= r(3,\lambda)\cdot 2\cdot 4 \\ &= \lambda\cdot 2\cdot 4 \\ &= 24 \end{aligned}$$

(ii)

$P(\omega)$: $r(x,r(y,\omega)) = r(y,r(x,\omega))$ $\qquad \forall x, y$

Basis step:

$P(\lambda)$: $r(x,r(y,\lambda)) = r(y,r(x,\lambda))$

$\Leftrightarrow \quad r(x,\lambda) = r(y,\lambda)$ $\qquad$ by defn of r

$\Leftrightarrow \quad \lambda = \lambda$ $\qquad$ by defn of r

$\Leftrightarrow \quad True$

Inductive step:

(i)

$x = y = t$

$P(\omega)$: $r(x,r(y,\omega)) = r(y,r(x,\omega))$

$\Leftrightarrow \quad r(t,r(t,\omega)) = r(t,r(t,\omega))$

$\Leftrightarrow \quad True$

(ii)

$x \neq t,\ y \neq t$

$P(\omega)$: $r(x,r(y,\omega)) = r(y,r(x,\omega))$

$\Leftrightarrow \quad r(x,r(y,\omega))\cdot t = r(y,r(x,\omega))\cdot t$

$\Leftrightarrow \quad r(x,r(y,\omega)\cdot t) = r(y,r(x,\omega)\cdot t)$ $\qquad$ by defn

$\Leftrightarrow \quad r(x,r(y,\omega\cdot t)) = r(y,r(x,\omega\cdot t))$ $\qquad$ by defn

$\Leftrightarrow \quad P(\omega\cdot t)$

(iii)

$x \neq t,\ y = t$

$P(\omega):\ r(x, r(y,\omega)) = r(y, r(x,\omega))$

$\Leftrightarrow\quad r(x, r(t,\omega)) = r(t, r(x,\omega))$

$\Leftrightarrow\quad r(x, r(t,\omega \cdot t)) = r(t, r(x,\omega))$ by defn

$\Leftrightarrow\quad r(x, r(t,\omega \cdot t)) = r(t, r(x,\omega)\cdot t)$ by defn

$\Leftrightarrow\quad r(x, r(t,\omega \cdot t)) = r(t, r(x,\omega \cdot t))$ by defn

$\Leftrightarrow\quad r(x, r(y,\omega \cdot t)) = r(y, r(x,\omega \cdot t))$

$\Leftrightarrow\quad P(\omega \cdot t)$