SECTION A Answer **all** questions in this section

Question A1

Use Quine's method, together with the standard laws of logical equivalence to show that the expression $P \wedge \big((P \wedge R) \vee (R \to (P \to Q))\big)$ is logically equivalent to P.

[10]

Question A2

With full explanation, state a model and countermodel for each of the following formulae in predicate logic.

a) $\forall x \cdot (P(x) \to Q(x))$

b) $\forall x \cdot \forall y \cdot (P(x, y) \to \neg P(x, y))$

[10]

Question A3

a) Use the generalised pigeonhole principle to show that, among any 91 integers, there are at least ten that end with the same digit.

[3]

b) Using the pigeonhole principle, show that every sequence of $n^2 + 1$ distinct integers contains a subsequence of length $n+1$ that is either strictly increasing or decreasing.

[5]

Suggestion: For each element of the sequence, a_k, associate an ordered pair (i_k, d_k) where, starting at a_k, i_k is the length of the longest increasing subsequence and d_k is the length of the longest decreasing subsequence. The above theorem can then be proven using proof by contradiction.

Question A4

a) Prove that a group table has to be a Latin square. You may wish to do this by means of proof by contradiction.

[6]

b) By using the fact that all group tables are Latin squares, explain why there are a maximum of one group of order three.
Note that groups that are isomorphic are considered identical.

[4]

Question A5

Consider a relation, *R*, on $A = \{a, b, c, d, e\}$, represented by the following matrix:

$$M_R = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

a) Show that *R* is an equivalence relation. [4]

b) State the equivalence classes of R. [1]

c) Give an example of a relation on *A* which is symmetric and reflexive but is not transitive. Explain why your selected relation is not transitive.

[5]

Question A6

Let Σ be the set of all bit strings including the empty string. Consider the set *B*, being a subset of Σ and defined recursively as follows:

Basis step:

$$0 \in B$$

Recursive step:

$$\left((\omega_1 \in B) \wedge (\omega_2 \in B)\right) \rightarrow \left(S_1(\omega_1, \omega_2) \in B\right)$$
$$\left((\omega_1 \in B) \wedge (\omega_2 \in B)\right) \rightarrow \left(S_2(\omega_1, \omega_2) \in B\right)$$

with the successor functions given as follows:

$S_1 : B \times B \rightarrow B$ given by $S_1(\omega_1, \omega_2) = \omega_1 \cdot \omega_2$

$S_2 : B \times B \rightarrow B$ given by $S_2(\omega_1, \omega_2) = \omega_1 \cdot 1 \cdot \omega_2$

a) Determine with explanation whether the following bit strings are elements of *B*:
(i) 100 (ii) 0010 [3]

b) Show that the set *B* is not freely generated by this recursive definition. [3]

c) Prove by structural induction that every element of *B* will have more zeros than ones. [6]

Section B Answer **two** questions from this section

Question B1

a) Using laws of inference and logical equivalences, prove the validity of the following argument:

$$\{\neg(C \rightarrow D), C \rightarrow E, E \rightarrow (G \vee D)\} \vdash G$$

[9]

b)

(i) By obtaining an interpretation that is a model for $\forall x \cdot P(x) \rightarrow \forall x \cdot Q(x)$ and a countermodel for $\forall x \cdot (P(x) \rightarrow Q(x))$, prove that the argument $\{\forall x \cdot P(x) \rightarrow \forall x \cdot Q(x)\} \vdash \forall x \cdot (P(x) \rightarrow Q(x))$ is invalid.

[4]

(ii) Using laws of inference and logical equivalences, prove the validity of the following argument:
$\{\forall x \cdot (P(x) \rightarrow Q(x))\} \vdash \forall x \cdot P(x) \rightarrow \forall x \cdot Q(x)$.

[7]

Question B2

a)

i) Using the recursive definition for the Stirling number of the second kind in the formula sheet, evaluate $S(4,3)$ as an integer.

[3]

ii) State all the partitions of $\{a,b,c,d\}$ into three parts.

[2]

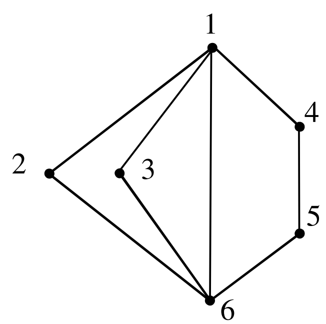
iii) State a surjection of the form $f: X \to W$ where $X = \{a,b,c,d\}$, $W = \{1,2,3\}$.

[2]

iv) By associating the surjection stated in part (iii) with a partition in part (ii), explain why the number of surjections of the form $f: X \to W$ is equal to $3!S(4,3)$.

[4]

b) Consider the following graph with the vertex set $V = \{1,2,3,4,5,6\}$:



Let G be the automorphism group for this graph.

i) Write down the elements of G as permutations of the vertices of the graph. [3]

ii) Determine the orbits of G on the vertex-set V. [1]

iii) For the tree above, use Burnside's lemma to find the number of essentially distinct ways in which it can be coloured using four colours.
Here two tree colourings are taken as equivalent if there is an automorphism of the tree that takes one tree colouring to the other.

[5]

Note: You may wish to use Burnside's lemma for part (iii), as supplied in the formula sheet.

Question B3

a) Solve the following second order recurrence relation:

$$a_n - 4a_{n-1} + 3a_{n-2} = 4,$$

with initial conditions $a_0 = 1,\ a_1 = 3$.

[10]

b) Let $\sum_n$ be the set of all strings of length *n* on the alphabet $A = \{0,1,2,3,4\}$.

Let S_n be the set of all elements of $\sum_n$ that have an even number of zeros.

(i) State the sets S_1, S_2. [2]

(ii) Let $a_n = |S_n|$. Obtain a first order recurrence relation for a_n. [7]

(iii) Verify that your result in part (*ii*) is consistent with your result of part (*i*). [1]

Question B4

a) Prove by mathematical induction that $(1+h)^n \geq 1+nh$ for all $n \in \mathbb{N}$ for the case of $h > 0$.

[5]

b) Consider the function $a:\mathbb{N} \to \mathbb{N}$ defined recursively by the following definition:

Basis step: $a(0)=0$

Recursive step: $a(n)=n-a(a(n-1))$

Evaluate $a(5)$.

[3]

d) Let $A=\{n \in \mathbb{N} \mid n \leq 9\}$ and let Σ be the set of all strings on A including the empty string, λ.
Consider the function $r: A \times \Sigma \to \Sigma$ defined recursively as follows:

Basis step:

$r(x,\lambda)=\lambda \qquad \forall x \in A.$

Recursive step:

$$r(x,\omega \cdot t)=\begin{cases} r(x,\omega) & \text{if } x=t \\ r(x,\omega)\cdot t & \text{if } x \neq t \end{cases} \qquad \forall x,t \in A,\ \forall \omega \in \Sigma.$$

(i) Evaluate $r(3,234)$. [3]

(ii) Use proof by structural induction to prove the following identity:
$r(x,r(y,\omega))=r(y,r(x,\omega)) \qquad \forall x,y \in A,\ \forall \omega \in \Sigma$

[9]

Hint:
Use the propositional function, $P(\omega)$: $r(x,r(y,\omega))=r(y,r(x,\omega))$.
In the induction step, in proving $P(\omega) \to P(\omega \cdot t)$ prove separately the cases of

i) $(x=t)\wedge(y=t)$
ii) $(x \neq t)\wedge(y \neq t)$
iii) $(x \neq t)\wedge(y=t)$

In each case, first simplify the expression $P(\omega \cdot t)$ using the recursive definition of the function r. Then, if necessary, assume $P(\omega)$ to arrive at $P(\omega \cdot t)$.