UFQEFW-20-2 DISCRETE MATHEMATICS

2005 SUMMER EXAMINATION SOLUTIONS

Question A1

$W(P/True)$	$\Leftrightarrow$	$(True \to Q) \wedge (Q \to R) \to (True \to R)$	
	$\Leftrightarrow$	$(False \vee Q) \wedge (Q \to R) \to (False \vee R)$	Implication Law (twice)
	$\Leftrightarrow$	$(Q \wedge (Q \to R)) \to R$	Identity Law (twice)*
	$\Leftrightarrow$	$(Q \wedge (\neg Q \vee R)) \to R$	Implication Law
	$\Leftrightarrow$	$((Q \wedge \neg Q) \vee (Q \wedge R)) \to R$	Distributive Law
	$\Leftrightarrow$	$(False \vee (Q \wedge R)) \to R$	Contradiction
	$\Leftrightarrow$	$(Q \wedge R) \to R$	Identity Law
	$\Leftrightarrow$	$\neg(Q \wedge R) \vee R$	Implication Law
	$\Leftrightarrow$	$(\neg Q \vee \neg R) \vee R$	DeMorgans Law
	$\Leftrightarrow$	$\neg Q \vee (\neg R \vee R)$	Associative Law
	$\Leftrightarrow$	$\neg Q \vee True$	Excluded Middle Law
	$\Leftrightarrow$	$True$	Domination Law

Majority of students expected to create a supplementary application of Quine’s from *. That is:

Let $Y \quad \Leftrightarrow \quad (Q \wedge (Q \to R)) \to R$

Show that Y is a tautology in order to show that $W(P/True)$ is a tautology.

$Y(Q/True)$	$\Leftrightarrow$	$(True \wedge (True \to R)) \to R$	
	$\Leftrightarrow$	$(True \to R) \to R$	identity law
	$\Leftrightarrow$	$(False \vee R) \to R$	implication law
	$\Leftrightarrow$	$R \to R$	identity law
	$\Leftrightarrow$	$\neg R \vee R$	implication law
	$\Leftrightarrow$	$True$	excluded middle law

$Y(Q/False)$	$\Leftrightarrow$	$(False \wedge (False \to R)) \to R$	
	$\Leftrightarrow$	$False \to R$	domination law
	$\Leftrightarrow$	$True \vee R$	implication law
	$\Leftrightarrow$	$True$	domination law

Since $Y(Q/True)$ and $Y(Q/False)$ are tautologies then so is $W(P/True)$.

$W(P/False)$	$\Leftrightarrow$	$(False \to Q) \wedge (Q \to R) \to (False \to R)$	
	$\Leftrightarrow$	$(True \vee Q) \wedge (Q \to R) \to (True \vee R)$	Implication Law (twice)
	$\Leftrightarrow$	$(True \wedge (Q \to R)) \to True$	Domination Law (twice)
	$\Leftrightarrow$	$(Q \to R) \to True$	Identity Law (not necessary)
	$\Leftrightarrow$	$\neg(Q \to R) \vee True$	Implication Law
	$\Leftrightarrow$	$True$	Domination Law

Since both of W(P/True) and W(P/False) are tautologies then we arrive at the conclusion.

Question A2

a)

$$\forall x \cdot \neg(P(x) \to Q(x))$$
$$D = \{1,\ 2,\ 3\}$$
$$P(x) = 'x < 4'$$
$$Q(x) = 'x \geq 4'$$

Numerous ways of doing this:

$$\forall x \cdot \neg(P(x) \to Q(x))$$
$$\Leftrightarrow \quad \forall x \cdot \neg(\neg P(x) \vee Q(x))$$
$$\Leftrightarrow \quad \forall x \cdot (P(x) \wedge \neg Q(x))$$

Truth values of P(1), P(2), P(3) are all True and
Truth values of Q(1), Q(2), Q(3) are all False.

Hence $P(x) \wedge \neg Q(x)$ is True for **<u>all</u>** of x = 1, 2, 3, and hence the result.

b)

The important thing about these questions is that the students *explain* their answer. A solution without an explanation will score less than half the marks available.

(i) <u>Model</u>:

$$\exists x \cdot \neg\big(P(x) \vee \neg Q(x)\big) \Leftrightarrow \quad \exists x \cdot \big(\neg P(x) \wedge Q(x)\big)$$

D = {1}
P(x): x = 2
Q(x): x = 1

Since there exists an x such that P(x) is false and Q(x) is true then we have a model for
$\exists x \cdot \big(\neg P(x) \wedge Q(x)\big)$

<u>Countermodel</u>:

$$\neg\exists x \cdot \neg\big(P(x) \vee \neg Q(x)\big) \quad \Leftrightarrow \quad \forall x \cdot \big(P(x) \vee \neg Q(x)\big)$$

For any universe of discourse, if we choose the propositional functions P and Q to be identical then we have a model for
$\forall x \cdot \big(P(x) \vee \neg Q(x)\big)$
and hence a countermodel for the original expression.

(ii) <u>Model</u>:

D = {1}
Q(x): x =2

Since the hypothesis of the implication is False then the overall expression must be true.

Countermodel:

To obtain a countermodel of the implication we find a model for the hypothesis and a countermodel for the conclusion.

D={1}
Q(x): x=1
P(x): *free choice!*

Hence we have a model for $\forall x \cdot \left(P(x) \to Q(x) \right)$ and thus a countermodel for the conclusion. We also have a model for the hypothesis and hence, overall, a countermodel.

Question A3

a)

Let n be the number of cards selected from the pack and they are distributed into pigeonholes corresponding to their suites. Hence there are four suites and at least one of the suites will have at least $\left\lceil \frac{n}{4} \right\rceil$ cards. We want this figure to be at least three. Hence $\left\lceil \frac{n}{4} \right\rceil \geq 3$ and the lowest possible value of n for which this holds is 9. Hence choosing 9 cards will guarantee that we'll have three from the same suite.

b)

(i) $|X| = n+1, |Y| = n$. Consider the function as allocating the elements of X into the pigeonholes, these being the elements of Y.

Since there are n+1 objects and n pigeonholes then, by the pigeonhole principle, there will exist an element of Y allocated to two elements of X. Hence the function is not an injection.

(ii) Since f cannot be an injection then there will be two elements of X of the following form: $x_1 = q \times 2^{k_1}$, $x_2 = q \times 2^{k_2}$. That is, they share the same highest odd divisor. With no loss of generality, we assume that $k_2 > k_1$.

Hence $x_2 / x_1 = 2^{k_2 - k_1}$ which is an integer. Hence x_1 divides x_2.

Question A4

Let $G = \{(x, y, z) \in \mathbb{R}^3 \mid (x = 1) \wedge (z = 0)\}$.

Show that G is a group under the following binary operation:

$(x_1, y_1, z_1) \circ (x_2, y_2, z_2) \equiv (x_1x_2 + y_1z_2,\ x_1y_2 + y_1,\ z_1x_2 + z_2)$

$G = \left\{(1, y, 0) \in \mathbb{R}^3 \mid True\right\}$

Closure:

$(1, y_1, 0) \in G$
$(1, y_2, 0) \in G$
$(1, y_1, 0) \circ (1, y_2, 0) = (1 \cdot 1 + y_1 \cdot 0,\ 1 \cdot y_2 + y_1,\ 0 \cdot 1 + 0) = (1,\ y_2 + y_1,\ 0) \in G$
Satisfied.

Associativity:

$(1, y_1, 0) \circ (1, y_2, 0) = (1,\ y_2 + y_1,\ 0)$
$\left((1, y_1, 0) \circ (1, y_2, 0)\right) \circ (1, y_3, 0) = (1,\ \left(y_2 + y_1\right) + y_3,\ 0)$
$(1, y_1, 0) \circ \left((1, y_2, 0) \circ (1, y_3, 0)\right) = (1,\ y_2 + \left(y_1 + y_3\right),\ 0)$

Due to associativity of addition, the above expressions are identical hence satisfied.

Identity:

$e = (1, 0, 0) \in G$
$(1, y, 0) \circ (1, 0, 0) = (1,\ y,\ 0)$
$(1, 0, 0) \circ (1, y, 0) = (1,\ y,\ 0)$

Satisfied

Inverse:

$(1, y, 0)^{-1} = (1, -y, 0) \in G$
$(1, y, 0) \circ (1, -y, 0) = (1,\ y - y,\ 0) = e$

Satisfied

Question A5

a) $\{1,\ 100, 101\} \subseteq C$ from the basis step

Applying the successor function to 1 or 101 results in 1 or 101, respectively. Hence the successor function does not produce new elements from two of the elements in the basis set.

The only way new elements are generated by S is to successively apply it to 100.

S(1) = 1	(no progress)
S(100) = 10021002	(new element)
S(101) = 101	(no progress)

S(100)	= 10021002
S°S(100)	= 010021002
S°S(100)	= 0010021002
S°S°S°S(100)	= 00010021002

b) The only way a successor function can return a string with a 1 at the end of it is if the input had a 1 at the end of it. Even then, the successor function doesn't generate a new element as it simply returns the input unaltered. Hence no new element of C ending in a 1 can be generated from the recursive step. Thus, any element of C that ends in a 1 must have originated from the basis step.

Question A6

a)

$$R=\{(a,a),(a,b),(b,a),(b,b)\}=A^2$$
$$R=\{(a,b),(b,a),(b,b)\}$$
$$R=\{(a,a),(a,b),(b,a)\}$$
$$R=\{(a,b),(b,a)\}$$
$$R=\{(a,a),(b,b)\}$$
$$R=\{(b,b)\}$$
$$R=\{(a,a)\}$$
$$R=\{\ \}=\varnothing$$

b)

(i) Given a symmetric relation on a set of cardinality n. Consider the matrix representation of this symmetric relation and denote this matrix by M_n. Consider the matrix representation of a relation on a set with cardinality $n+1$ that has the aforementioned symmetric relation in its top right hand corner, viz,

$$\begin{pmatrix} \boxed{\begin{matrix} & & \\ & M_n & \\ & & \end{matrix}} & \begin{matrix} x \\ x \\ x \\ x \\ x \end{matrix} \\ x \quad x \quad x \qquad x \quad x & x \end{pmatrix}$$

where the values marked by the x's are, as yet, unspecified. For the M_{n+1} to be symmetric we require that the $n+1^{\text{st}}$ row and column are identical. Hence we have a choice of $n+1$ entries, occupied by zeros or ones. The number of possible options is therefore 2^{n+1}.

Since the M_n can represent any symmetrical relation it must be the case, by the product rule, that $M_{n+1}=2^{n+1}\times M_n$.

(ii) Define the propositional function: $P(n): \quad S_n = 2^{(n^2+n)/2}$

Basis step

The symmetric relations on a singleton set $\{a\}$ is two, that is,

$R = \{(a,a)\} = A^2$

$R = \{\ \} = \varnothing$

and this is consistent with the theorem as: $2^{(1^2+1)/2} = 2$

Inductive step

Prove that $P(n) \Rightarrow P(n+1)$.

$$
\begin{aligned}
P(n) \quad &\Rightarrow \quad 2^{n+1} \times S_n = 2^{n+1} \times 2^{(n^2+n)/2} \\
&\Rightarrow \quad 2^{n+1} \times S_n = 2^{n+1} \times 2^{(n^2+n)/2} \\
&\Rightarrow \quad S_{n+1} = 2^{n+1} \times 2^{(n^2+n)/2} \qquad \text{since } S_{n+1} = 2^{n+1} \times S_n \\
&\Rightarrow \quad S_{n+1} = 2^{(n^2+3n+2)/2} \\
&\Rightarrow \quad S_{n+1} = 2^{((n+1)^2+(n+1))/2} \\
&\Rightarrow \quad P(n+1)
\end{aligned}
$$

Question B1

a) $P(n): \ \sum_{k=1}^{n}(-1)^{k-1}k^2 = (-1)^{n-1}n(n+1)/2$

Basis step:

$$P(1): \quad \sum_{k=1}^{1}(-1)^{k-1}k^2 = (-1)^{1-1}1(2)/2$$
$$\Leftrightarrow \quad (-1)^{1-1}1^2 = (-1)^{1-1}1(2)/2$$
$$\Leftrightarrow \quad 1 = 1$$
$$\Leftrightarrow \quad \textit{True}$$

Basis step is satisfied.

Inductive step:

$P(n) \Rightarrow P(n+1)$

$$P(n) \quad \Rightarrow \quad \sum_{k=1}^{n}(-1)^{k-1}k^2 = (-1)^{n-1}n(n+1)/2$$
$$\Rightarrow \quad (-1)^n(n+1)^2 + \sum_{k=1}^{n}(-1)^{k-1}k^2 = (-1)^n(n+1)^2 + (-1)^{n-1}n(n+1)/2$$
$$\Rightarrow \quad \sum_{k=1}^{n+1}(-1)^{k-1}k^2 = (-1)^n(n+1)^2 + (-1)^{n-1}n(n+1)/2$$
$$\Rightarrow \quad \sum_{k=1}^{n+1}(-1)^{k-1}k^2 = (-1)^n\left((n+1)^2 - n(n+1)/2\right)$$
$$\Rightarrow \quad \sum_{k=1}^{n+1}(-1)^{k-1}k^2 = (-1)^n\left(n^2+3n+2\right)/2$$
$$\Rightarrow \quad \sum_{k=1}^{n+1}(-1)^{k-1}k^2 = (-1)^n(n+1)(n+2)/2$$
$$\Rightarrow \quad P(n+1)$$

b) (i) $A = \{0,1,2\}$

(ii)

$f(0) = 1$

$f(1) = f(0) + 1 = 2$

$f(2) = f(1) + 1 = 3$

$f(0) = f(2) + 1 = 4$

Not well defined since $f(0)$ is multi-valued.

$g(0) = 1$

$g(1) = \left(g(0)\right)^2 = 1$

$g(2) = \left(g(1)\right)^2 = 1$

$g(0) = \left(g(2)\right)^2 = 1$

g is a well defined function.

c) (i)

$$\begin{aligned} r(100) &= r(10\cdot 0) = 0\cdot r(10) = 0\cdot r(1\cdot 0) = 0\cdot(0\cdot r(1)) \\ &= 0\cdot(0\cdot r(l\cdot 1)) = 0\cdot(0\cdot(1\cdot r(l))) = 0\cdot(0\cdot(1\cdot l)) = 001 \end{aligned}$$

(ii) Define $P(w)$: $r(100\cdot w) = r(w)\cdot 001$

Basis step:

$r(100\cdot l) = r(100) = 001$

$r(l)\cdot 001 = l\cdot 001 = 001$

Hence $r(100\cdot l) = r(l)\cdot 001$

Hence $P(l)$ is true

Inductive step:

$P(w\cdot 1)$

$\Leftrightarrow$	$r(100\cdot w\cdot 1) = r(w\cdot 1)\cdot 001$	by definition of $P(w\cdot 1)$
$\Leftrightarrow$	$1\cdot r(100\cdot w) = 1\cdot r(w)\cdot 001$	by defn of r
$\Leftrightarrow$	$r(100\cdot w) = r(w)\cdot 001$	left cancellation
$\Leftrightarrow$	$P(w)$	by definition of $P(w)$

Since $P(w\cdot 1) \Leftrightarrow P(w)$ it must be the case that $P(w) \Rightarrow P(w\cdot 1)$ and the inductive step is satisfied for $w\cdot 1$.

Similar approach results in $P(w) \Rightarrow P(w\cdot 0)$.

Question B2

a) AAAABBBCCD

Four tasks in sequence: chose the position of the As, Bs, Cs and Ds in sequence.

Positions of the As = C(10,4)
Positions of the Bs = C(6,3)
Positions of the Cs = C(3,2)
Positions of the Ds = C(1,1)

Product rule results in C(10,4)×C(6,3)×C(3,2)×C(1,1) = 12600

b) $W = \{a, b, c\}, \quad X = \{a, b, c, d\}$

i) A surjection of the form $f : X \to X$ will be a bijection and vice versa. Number of such bijections is 4! = 24.

ii) Surjections of the form $f : X \to W$.

The number of surjections of the form $f : X \to W$

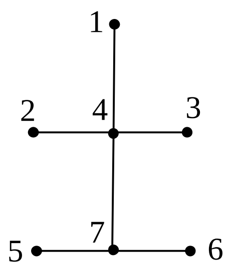
= the number of partitions of X into three parts × the number of permutations of W

$= 3! \times S(4,3)$

$$
\begin{aligned}
&= 6 \times S(4,3) \\
&= 6 \times \left(S(3,2) + 3S(3,3)\right) \\
&= 6 \times \left(S(2,1) + 2S(2,2) + 3\right) \\
&= 6 \times (1 + 2 + 3) \\
&= 36
\end{aligned}
$$

i)

$G = \{(),$
$(12),\ (23),\ (13),\ (123),\ (132),$
$(56),$
$(12)(56),\ (23)(56),\ (13)(56),\ (123)(56),\ (132)(56)\ \}$



ii) Orbits = $\{\{1,2,3\},\{4\},\{7\},\{5,6\}\}$

iii)

$$|fix()| = 4^7$$
$$|fix(12)| = |fix(23)| = |fix(13)| = 4^6$$
$$|fix(123)| = |fix(132)| = 4^5$$
$$|fix(56)| = 4^6$$
$$|fix(12)(56)| = |fix(23)(56)| = |fix(13)(56)| = 4^5$$
$$|fix(123)(56)| = |fix(132)(56)| = 4^4$$

$$\frac{1}{|G|}\sum_{s\in G} |\ fix(s)\ | = \frac{1}{12}\left(4^7 + \left(4^6+4^6+4^6+4^6\right) + \left(4^5+4^5+4^5+4^5+4^5\right) + \left(4^4+4^4\right)\right)$$
$$= \frac{4^4\left(4^3+\left(4^2+4^2+4^2+4^2\right)+\left(4+4+4+4+4\right)+\left(1+1\right)\right)}{12} = 3200$$

Question B3

1.	$(A \wedge B) \vee C$	Premise
2.	$D \to \neg C$	Premise
3.	$(B \vee C) \to E$	Premise
4.	D	Assumption for $\to\_I$
5.	$\neg C$	$2,4,\to\_E(MP)$
6.	$A \wedge B$	$1,5,\vee\_E$
7.	A	$6,\wedge\_E$
8.	B	$6,\wedge\_E$
9.	$B \vee C$	$8,\vee\_I$
10.	E	$3,9,\to\_E(MP)$
11.	$E \wedge A$	$10,7,\wedge\_I$
12.	$D \to (E \wedge A)$	$4,11,\to\_I$

b) Consider the following <u>invalid</u> argument:

$$\{\ \forall x \cdot P(x) \to Q,\ P(c)\} \big/ Q$$

where P is a propositional function, Q is a proposition and c is an element in the universe of discourse.

(i) Domain: {1,2}
P(x): x = 1
Q: False
c = 1

(ii) The error in the proof is that the scope of the universal quantifier in line 1 is not the whole of the expression, that is, it is <u>not</u> the case that:

$$\forall x \cdot P(x) \to Q \quad \Leftrightarrow \quad \forall x \cdot (P(x) \to Q)$$

Hence, we're not allowed to apply $\forall\_E$ to line 1 as the quantifier's scope is not the whole of the expression. That is, line 3 is incorrect.

Had line 3 been the following:

$$\forall x \cdot (P(x) \to Q)$$

then applying the $\forall\_E$ inference rule would have been acceptable.

Question B4

a) Consider the following recurrence relation

$$a_n = a_{n-2} + (-1)^n$$

with initial conditions $a_0 = 0,\ a_1 = 0$.

Obtain a closed form for a_n.

Homogeneous case:

$$a_n^h = a_{n-2}^h$$

$\Rightarrow \quad r^n = r^{n-2}$

$\Rightarrow \quad r^2 = 1$

$\Rightarrow \quad r = \pm 1$

$\Rightarrow \quad a_n^h = a + b(-1)^n$

where a and b are arbitrary constants.

$$f(n) = (\text{polyomial in } n \text{ of order zero})s^n$$

$\Rightarrow$ the polynomial is a constant and $s = -1$ which is a root of the characteristic equation

$\Rightarrow \quad s = -1, t = 0, m = 1$

$\Rightarrow \quad a_n^{(p)} = nc(-1)^n$

where c is a (non-arbitrary) constant, dependent upon the recurrence relation.

The particular solution satisfies the recurrence relation. Use this information to obtain c.

$a_n^{(p)} = a_{n-2}^{(p)} + (-1)^n$

$\Rightarrow \quad nc(-1)^n = (n-2)c(-1)^{n-2} + (-1)^n$

$\Rightarrow \quad nc = (n-2)c + 1$

$\Rightarrow \quad c = 0.5$

Hence: $a_n = a + (b + 0.5n)(-1)^n$

Implement initial conditions to evaluate arbitrary constants:

$a_n = a + (b + 0.5n)(-1)^n$

$$\Rightarrow \begin{cases} a_0 = a + b = 0 \\ a_1 = a - (b + 0.5) = 0 \end{cases} \quad \Rightarrow \quad 2a - 0.5 = 0 \quad \Rightarrow \quad \begin{cases} a = 0.25 \\ b = -0.25 \end{cases}$$

Hence

$$\begin{aligned} a_n &= a + (b + nc)(-1)^n \\ &= 0.25\left(1 + (-1 + 2n)(-1)^n\right) \end{aligned}$$

b) Consider the Tower of Hanoi problem with three pegs and n discs. Let H_n be the number of moves required to solve the Tower of Hanoi problem.

i)

$H_1 = 1$

since there is only one move required to move the singleton disk from C1 to C3.

$H_2 = 3$

Move 1: D1 to C3
Move 2: D2 to C2
Move 3: D1 to C2

[3]

ii) State, with explanation, a recurrence relation for H_n.

We can move the n disks from C1 to C2 as follows:
Block move $n-1$ disks to C3 (that takes H_{n-1} moves).
Move the largest disk to C2.
Block move $n-1$ disks to C2 (that takes H_{n-1} moves).

Total number of moves = $2H_{n-1}+1$. Hence the recurrence relation is:

$H_n = 2H_{n-1}+1$

[3]

iii)

Homogeneous case:

$r^n = 2r^{n-1} \quad \Rightarrow \quad r = 2$

$H_n^{(h)} = a2^n$

where a is an arbitrary constant dependent upon the initial condition.

Particular solution:

$H_n^{(p)} = b$

where b is a (non-arbitrary) constant dependent upon the recurrence relation.

The particular solution satisfies the recurrence relation. Use this information to obtain b.

$H_n^{(p)} = 2H_{n-1}^{(p)}+1$
$\Rightarrow b = 2b+1$
$\Rightarrow b = -1$

Hence: $H_n = a2^n - 1$

Implement initial conditions to evaluate arbitrary constant:

$H_1 = 1 = a2 - 1 \quad \Rightarrow \quad a = 1$

Hence $H_n = 2^n - 1$.

Verification:

$H_n = 2^n - 1$

$H_{n-1} = 2^{n-1} - 1$

$2H_{n-1} + 1 = 2\left(2^{n-1} - 1\right) + 1 = 2^n - 2 + 1 = 2^n - 1 = H_n$

Hence satisfied.