UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics B45: Discrete Mathematics for Computer Scientists

COURSE CODE	:	**MATHB045**
UNIT VALUE	:	**0.50**
DATE	:	**09-MAY-05**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

05-C0914-3-90

TURN OVER

All questions may be attempted but only marks obtained on the best **five** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Show using a truth table that for any sets X, Y and Z,

$$(X \cup Y) \cap (Y \cup Z) = Y \cup (X \cap Z)$$

Shade the set $(X \cup Y) \cap (Y \cup Z)$ on a Venn diagram.

(b) Explain what it means for a set to be countable.

Show that $\mathbb{R}$ is uncountable.

2. (a) Define the terms *injective*, *surjective* and *bijective*.

(b) Let X and Y be non-empty sets and let $f : X \to Y$ be an injective function. Prove that f has a left-inverse.

(c) Give an example of a function $f : \mathbb{N} \to \mathbb{N}$, which is surjective but not injective.

Write down a right inverse of your function.

(d) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 3x + 1 & \text{if } x > 0, \\ 2 - x^2 & \text{if } x \leq 0. \end{cases}$$

Find a right-inverse of f.

3. (a) Consider the following permutations:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 3 & 4 & 5 & 1 & 6 & 8 & 10 & 7 & 9 \end{pmatrix},$$

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 5 & 4 & 7 & 1 & 6 & 2 & 10 & 9 & 3 & 8 \end{pmatrix}.$$

Calculate $\sigma\tau$, $\tau\sigma$ and σ^{-1}.

Write σ and τ in disjoint cycle notation.

Find the orders of σ and τ.

Calculate σ^{34} and τ^{2004}, writing your answers in functional notation.

Calculate $\text{sign}(\sigma)$ and $\text{sign}(\tau)$.

Find an element of S_{10} with order 21.

(b) Prove that for any permutations $\sigma, \tau \in S_n$,

$$\text{sign}(\sigma\tau) = \text{sign}(\sigma)\text{sign}(\tau).$$

4. (a) State the axioms of a group.

State Lagrange's Theorem.

Hence show that for any element g of a finite group G, the order of g is a factor of $|G|$.

(b) List the elements of the dihedral group of D_{10}.

Find the order of each element.

Find all subgroups of D_{10}.

5. (a) Solve the congruence:

$$8x \equiv 3 \ (143).$$

(b) Define the Euler totient function φ and calculate $\varphi(143)$.

(c) Hence calculate 8^{319} modulo 143.

(d) Solve the congruence

$$y^{103} \equiv 2 \ (143).$$

6. (a) Find all real solutions to the following simultaneous equations:

$$\begin{array}{ccccccccc} w & + & 2x & + & 3y & + & 4z & = & 10, \\ w & + & x & + & y & + & z & = & 4, \\ & & 2x & + & 3y & & & = & 5. \end{array}$$

(b) Define the *determinant* of an $n \times n$ matrix $(a_{i,j})$.

(c) Calculate (i) the determinant, and (ii) the inverse of the following matrix:

$$\begin{pmatrix} 6 & 5 & 3 \\ 2 & 2 & 1 \\ 2 & 2 & 2 \end{pmatrix}.$$

7. (a) Find the eigenvalues and eigenvectors of the following matrices:

$$\begin{pmatrix} 11 & 18 \\ -6 & -10 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 & -1 \\ -1 & 2 & -1 \\ 3 & 1 & 4 \end{pmatrix}.$$

(b) Hence or otherwise show that for $n \in \mathbb{N}$,

$$\begin{pmatrix} 11 & 18 \\ -6 & -10 \end{pmatrix}^n = (-1)^n \begin{pmatrix} -3 & -6 \\ 2 & 4 \end{pmatrix} + 2^n \begin{pmatrix} 4 & 6 \\ -2 & -3 \end{pmatrix}.$$