UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics B45: Discrete Mathematics for Computer Scientists

COURSE CODE	:	**MATHB045**
UNIT VALUE	:	**0.50**
DATE	:	**19-MAY-04**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

04-C0910-3-120

TURN OVER

All questions may be attempted but only marks obtained on the best **five** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Show using a truth table that for any sets X, Y and Z,

$$\Big((X \cup Y) \cap (Y \cup Z)\Big) \cup X = X \cup Y.$$

Hence shade the set $\Big((X \cup Y) \cap (Y \cup Z)\Big) \cup X$ on a Venn diagram.

(b) Explain what it means for a set to be countable.

Let A and B be two countable sets. Show that $A \times B$ is countable.

Hence show that $\mathbb{Q}$ is countable.

2. (a) Define the terms *left inverse*, *right inverse* and *2-sided inverse* of a function $f : X \to Y$.

Define the terms *injective*, *surjective* and *bijective*.

(b) Let X and Y be non-empty sets and let $f : X \to Y$ be a surjective function. Prove that f has a right inverse.

(c) Give an example of a function $f : \mathbb{N} \to \mathbb{N}$, which is injective but not surjective.

Write down a left inverse of your function.

(d) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 2x + 1 & \text{if } x > 0, \\ 3 - x^2 & \text{if } x \leq 0. \end{cases}$$

Find a right-inverse of f.

3. (a) Consider the following permutations:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 3 & 4 & 5 & 1 & 6 & 8 & 10 & 7 & 9 \end{pmatrix},$$

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 5 & 4 & 7 & 1 & 6 & 2 & 10 & 9 & 3 & 8 \end{pmatrix}.$$

Calculate $\sigma\tau$, $\tau\sigma$ and σ^{-1}.

Write σ and τ in disjoint cycle notation.

Find the orders of σ and τ.

Calculate σ^{34} and τ^{2004}, writing your answers in functional notation.

Find an element of S_{10} with order 21.

(b) Define the *sign* of a permutation.

Prove that for any permutations $\sigma, \tau \in S_n$,

$$\text{sign}(\sigma\tau) = \text{sign}(\sigma)\text{sign}(\tau).$$

4. (a) State and prove Lagrange's Theorem.

(b) List the elements of the dihedral group D_8.

Find the order of each element.

Find all the cyclic subgroups of D_8.

5. (a) Solve the congruence:

$$23x \equiv 12 \bmod 95.$$

(b) Define the Euler totient function φ and calculate $\varphi(95)$.

(c) Hence calculate $4^{182} \bmod 95$.

(d) Solve the congruence

$$y^{29} \equiv 93 \bmod 95.$$

6. (a) Find all real solutions to the following simultaneous equations:

$$\begin{array}{rcrcrcrcl} w & + & x & + & 2y & + & z & = & 6, \\ 2w & + & x & + & y & + & z & = & 7, \\ 3w & + & x & + & y & & & = & 8. \end{array}$$

(b) Give an LU decomposition of the following matrix:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 2 & 1 \\ 2 & 0 & 1 & 0 \end{pmatrix}.$$

(c) Calculate (i) the determinant, and (ii) the inverse of the following matrix:

$$\begin{pmatrix} 4 & 2 & 1 \\ 4 & 4 & 3 \\ 1 & 1 & 1 \end{pmatrix}.$$

7. (a) Define the terms *eigenvalue* and *eigenvector* of an $n \times n$ matrix.

(b) Find the eigenvalues and eigenvectors of the following matrices:

$$\begin{pmatrix} 13 & -12 \\ 4 & -3 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 & -1 \\ -1 & 2 & -1 \\ 3 & 1 & 4 \end{pmatrix}.$$

(c) Find a matrix A such that

$$A^2 = \begin{pmatrix} 13 & -12 \\ 4 & -3 \end{pmatrix}.$$

END OF PAPER